MATH 532, 736I: MODERN GEOMETRY

Name ______________________________

Final Exam (1995)

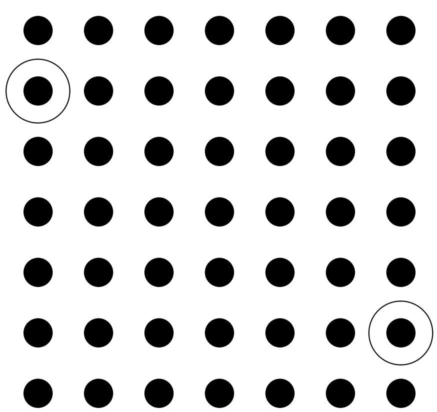
Show All Work

There are 200 total points on the Exam. The first 100 points are related to the first half of the course, and the second 100 points to the second half of the course. The value of each problem is indicated below.

Problem 1	14 points
Problem 2	10 points
Problem 3	18 points
Problem 4	18 points
Problem 5 (a)	6 points
Problem 5 (b)	8 points
Problem 5 (c)	6 points
Problem 5 (d)	8 points
Problem 5 (e)	12 points
Problem 6	20 points
Problem 7	20 points
Problem 8	20 points
Problem 9	20 points
Problem 10	20 points

(1) State the axioms for a finite projective plane of order n.

(2) Two points have been circled in the 5×5 array of points below. Using the model for a finite affine plane of order 5 discussed in class, finish circling the points that belong to the same line as the given circled points.



(3) Consider the points $(14, 6)$ and $(8, 11)$ in a 17×17 array of points for our model of a finite affine plane of order 17. Find the equation of the line passing through these two points. Put your answer in the form $y \equiv mx + k \pmod{17}$ where m and k are among the numbers $0, 1, 2, \ldots, 16$.

$m = \boxed{}$

$k = \boxed{}$

(4) Using only the axioms stated on the last page of this test (and referring to them whenever you use them), prove that if ℓ is a line in an affine plane of order n with exactly n points on it and A is a point not on ℓ, then there are exactly $n+1$ lines passing through A.

(5) The following problems (parts (a) through (e)) all deal with an axiomatic system consisting of the axioms below.

Axiom 1. There exist at least 2 distinct lines.

Axiom 2. Each line has exactly 2 points on it.

Axiom 3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom 4. Given any line ℓ and any point P not on ℓ, there are at least 2 distinct lines through P that do not intersect ℓ.

(a) Explain why the axiomatic system is consistent. (Comment: If you get stuck, you may want to look ahead at part (e).)

(b) Explain why the axiomatic system is independent.

(c) Explain why the axiomatic system is NOT complete.

(d) Does the principle of duality hold? Justify your answer.

(e) Prove that there are at least 5 distinct points no 3 of which are collinear.

(6) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 3 using Theorem 1 (but not Theorem 2 or Theorem 4).

(7) On the third to the last page is a figure consisting of 2 triangles ΔABC and $\Delta A'B'C'$ which are perspective from the line passing through X, Y, and Z. Suppose we want to use Desargues' Theorem to prove that these triangles are also perspective from a point. Let D be the point of intersection of the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. To show that $\overleftrightarrow{CC'}$ also passes through D, we can consider two triangles which we know to be perspective from a point. The vertices of these two triangles (not necessarily the edges) and the point they are perspective from are all in the figure. Determine the two triangles and the point they are perspective from and indicate your answers in the boxes below. (No work necessary.)

Triangles $\boxed{}$ and $\boxed{}$ are perspective from point $\boxed{}$.

(Note: You can complete the boxes to make a sentence which is true, but that will not necessarily insure that you have a correct answer. A correct answer must correspond to establishing that $\overleftrightarrow{CC'}$ passes through D as discussed above.)

(8) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ intersect at some point R. The next two pages contain a proof that P, Q, and R are collinear except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that P, Q, and R are collinear.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{\quad\quad}$.

Give explanation here: $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{\qquad\qquad}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$.

(Your answer should contain Q, B, and C.)

Also, from $k_2C - A = k_2C' - A'$, we deduce that

(3) $\boxed{}$.

(Your answer should contain R, A, and C.)

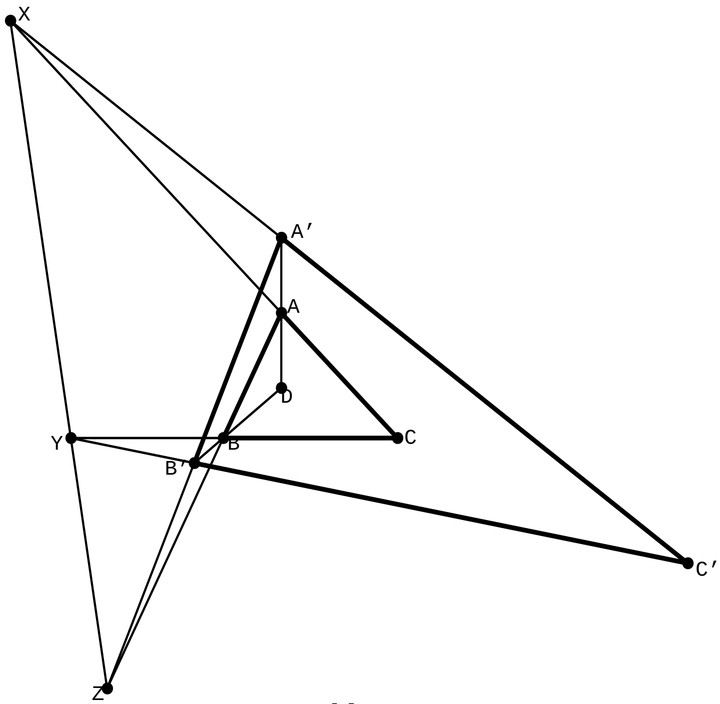
Therefore, from (1), (2), and (3),

$$(1 - k_1)P + (k_1 - k_2)Q + (k_2 - 1)R = \overrightarrow{0}.$$

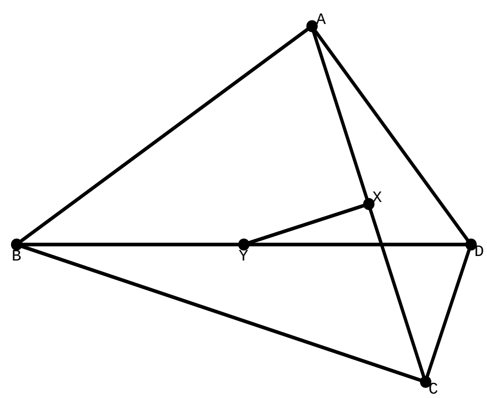
The result follows from Theorem $\boxed{}$ (on the last page of this test).

(9) A quadrilateral has 4 vertices A, B, C, and D oriented counter-clockwise (a picture for this problem is on the third to the last page of the test). Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$ are perpendicular and that $\overleftrightarrow{CB}$ and $\overleftrightarrow{CD}$ are perpendicular. Let X be the midpoint of $\overline{AC}$, and let Y be the midpoint of $\overline{BD}$. Using vectors, prove that $\overleftrightarrow{XY}$ is perpendicular to $\overleftrightarrow{AC}$. (You must use vectors appropriately to obtain credit for this problem.)

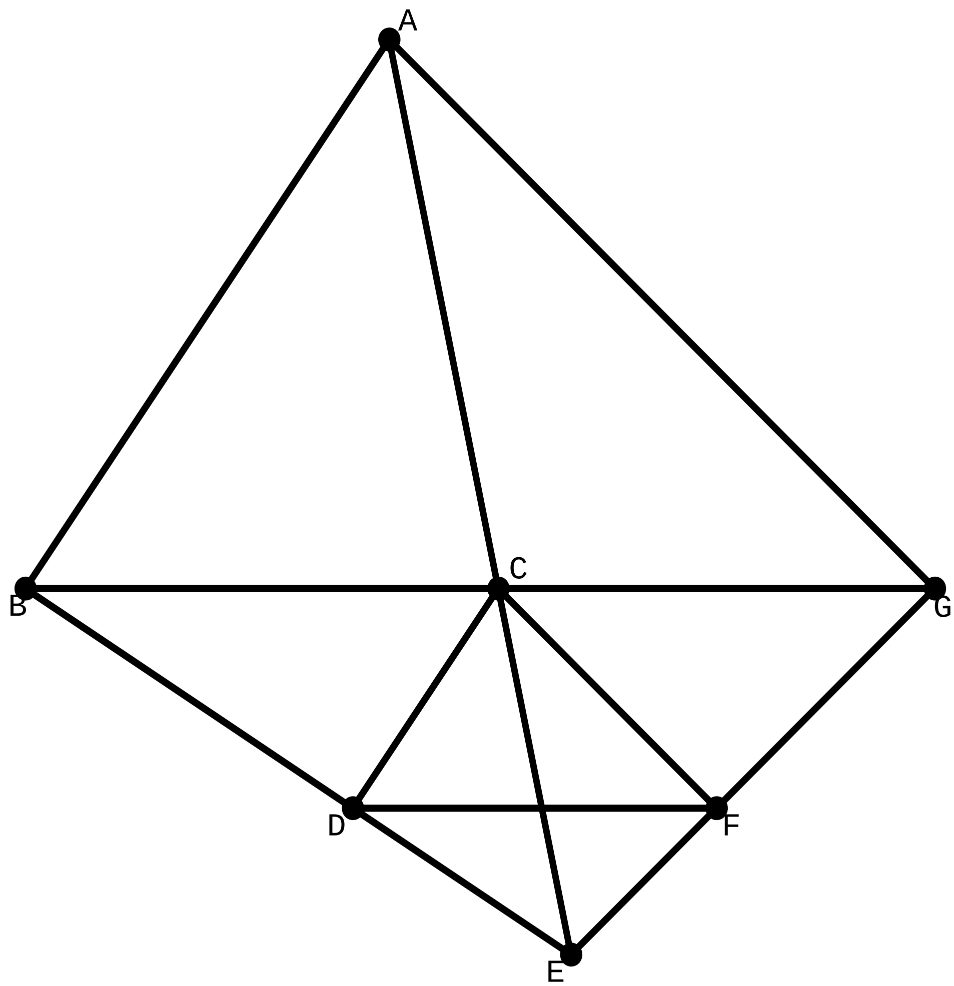
(10) In the figure on the second to the last page of this test, points A, C, and E are collinear, points B, C, and G are collinear, line $\overleftrightarrow{AG}$ is parallel to line $\overleftrightarrow{CF}$, and line $\overleftrightarrow{BG}$ is parallel to line $\overleftrightarrow{DF}$. Explain why Desargues' Theorem implies that line $\overleftrightarrow{AB}$ is parallel to line $\overleftrightarrow{CD}$. Desargues' Theorem refers to triangles perspective from a point and from a line. In your explanation, you should indicate clearly what the triangles are, what point they are perspective from, and what line they are perspective from. Some points and lines may be at "infinity" and, hence, may not appear in the figure.



Problem 7



Problem 9



Problem 10

INFORMATION PAGE

Axiom A1. There exist at least 4 distinct points no 3 of which are collinear.

Axiom A2. There exists at least 1 line with exactly n points on it.

Axiom A3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom A4. Given any line ℓ and any point P not on ℓ, there is exactly 1 line through P that does not intersect ℓ.

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then A, B, and C are collinear.

Theorem 4: *If A, B, and C are not collinear and if there are real numbers x, y, and z such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then $x = y = z = 0$.