A Theorem Concerning Affine Planes

Theorem: *In an affine plane of order n, each point has exactly $n+1$ lines passing through it.*

Lemma. *If ℓ is a line with exactly n points on it (in a finite affine plane of order n) and A is a point not on ℓ, then there are exactly $n+1$ lines passing through A.*

Proof. Consider an ℓ with exactly n points on it and a point A not on ℓ. Let $P_1, \ldots, P_n$ be the points on ℓ. By Axiom A3, for each $j \in \{1, 2, \ldots, n\}$, there exists a line ℓ_j passing through A and P_j. Also, by Axiom A3, these lines are distinct (otherwise, there would be 2 distinct lines passing through 2 distinct P_j's, namely the line ℓ and a line passing through A). By Axiom A4, there is a line ℓ_{n+1} parallel to ℓ passing through A. Since each of $\ell_1, \ldots, \ell_n$ intersects ℓ, each of these n lines is different from the line ℓ_{n+1}. Thus, we have $n+1$ distinct lines passing through A. To show that there are exactly $n+1$ lines passing through A, we still need to show that there are no more lines passing through A. Let ℓ' be an arbitrary line passing through A. By Axiom A3, there is exactly one line passing through a point P_j on ℓ and the point A, namely ℓ_j. Thus, if ℓ' passes through some P_j, then $\ell' = \ell_j$. On the other hand, if ℓ' does not pass through some P_j, then ℓ' is parallel to ℓ. By Axiom A4, ℓ_{n+1} is the unique line passing through A and parallel to ℓ, so in this case $\ell' = \ell_{n+1}$. Therefore, there are exactly $n+1$ lines passing through A. ■

Lemma. *If ℓ is a line (in a finite affine plane of order n) and A is a point not on ℓ with exactly $n+1$ lines passing through it, then ℓ has exactly n points on it .*

Proof. By Axiom A4, exactly n of the lines passing through A intersect ℓ. By Axiom A3, each of these lines intersects ℓ in exactly one point (otherwise, there would be 2 distinct lines, namely ℓ and a line through A, passing through 2 distinct points on ℓ). Also, by Axiom A3, these points of intersection are distinct (otherwise, there would be 2 distinct lines passing through a point on ℓ and the point A). Thus, ℓ has n distinct points on it. Furthermore, there cannot be another point, say Q, on ℓ; otherwise, by Axiom A3, there would be another line passing through A and intersecting ℓ (namely at Q). Therefore, ℓ has exactly n distinct points on it. ■

Proof of Theorem. Let P be an arbitrary point. To prove the theorem, we now consider a line ℓ with n points on it (which exists by Axiom A2). If P is not on ℓ, then Lemma 1 implies that there are exactly $n+1$ lines passing through P. So suppose P is on ℓ. Let A, B, C, and D be the points which exist by Axiom A1 so that no 3 of these are collinear. Hence, at most 2 of these 4 points are on ℓ. By relabelling if necessary, we may suppose that A and B are not on ℓ. Since A, C, and D are not collinear, we deduce from Axiom A3 that there is a line ℓ_1 passing through A and C and a different line ℓ_2 passing through A and D. Since A, B, and C are not collinear and since A, B, and D are not collinear, the lines ℓ_1 and ℓ_2 do not pass through B. Also, by Axiom A3, there can be at most one line passing through A and P; thus, at least one of ℓ_1 and ℓ_2, call it ℓ', does not pass through P.

Recall that B is a point not on ℓ and ℓ has exactly n points on it, so by Lemma 1, we know that there are exactly $n+1$ lines passing through B. Since B is not one ℓ', we deduce now from Lemma 2 that there are exactly n points on ℓ'. Since P is not on ℓ', we deduce from another application of Lemma 1 that P must have exactly $n+1$ lines passing through it. This establishes the theorem. ■