T1. The number of times a prime $p$ divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$ (note the sum above is finite since $[n/p^k]$ is 0 if $k$ is large). We deduce that if $2^r$ is the highest power of 2 dividing $30!$, then $r = [30/2] + [30/4] + \cdots = 15 + 7 + 3 + 1 = 26$. Similarly, if $3^s$ is the highest power of 3 dividing $30!$, then $s = [30/3] + [30/9] + \cdots = 10 + 3 + 1 = 14$. Continuing, we obtain

$$30! = 2^{26} \times 3^{14} \times 5^7 \times 7^4 \times 11^2 \times 13^2 \times 17 \times 19 \times 23 \times 29.$$  

The number of divisors is obtained by multiplying one more than each of the exponents together, so the number of divisors is $27 \times 15 \times 8 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 = 2332800$.

T2. Draw line segments joining the $(0,0)$ to each of $(2\sqrt{3}, 2)$ and $(2\sqrt{3}, -2)$ and joining $(16,0)$ to each of $(11, 5\sqrt{3})$ and $(11, -5\sqrt{3})$. The answer is easily obtained by summing the appropriate areas of triangles and trapezoids and subtracting the sum of the areas of two sectors. The answer is $\sqrt{-8 + 80\sqrt{3} - 36\pi}$.

T3. The thisforlackofabettername points points are seen to be those of the form $(x,y)$ where $x \geq |y|$ and $x+y$ is even. The condition $x \geq |y|$ comes from noting that $(0,0)$ is the initial point of a path and other points are obtained by adding at least as much to the $x$-coordinate as the absolute value of what is added to the $y$-coordinate. The condition $x+y$ is even is seen, for example, by induction. That each of these points can be obtained is also easy to see (to get such an $(x,y)$, simply consider the path from $(0,0)$ to $((x-y)/2, (y-x)/2)$ along a straight line and then from $((x-y)/2, (y-x)/2)$ to $(x,y)$ along a straight line).

The number of such $(x,y)$ satisfying $0 \leq x \leq 10$ and $0 \leq y \leq 10$ is 36.

T4. Since $PA - PC = (PA - PB) + (PB - PC)$, it suffices to consider the case that only $PA - PB$ and $PB - PC$ are integers. The sum of the lengths of two sides of a triangle is at least as big as the length of the third side so that $PA + AB \geq PB$ (by considering $\Delta ABP$) and $PB + BC \geq PC$ (by considering $\Delta BCP$). Note that equality could occur if $A, B,$ and $P$ are collinear or if $B, C,$ and $P$ are collinear, respectively. Similarly, we get $PB + AB \geq PA$ and $PC + BC \geq PB$. The conditions in the problem imply $AB = 1$ and $BC = 2$. It follows that $-1 \leq PA - PB \leq 1$ and $-2 \leq PB - PC \leq 2$. Since $PA - PB$ and $PB - PC$ are integers, we have $PA - PB \in \{-1,0,1\}$ and $PB - PC \in \{-2,-1,0,1,2\}$. The condition $PA - PB = \pm 1$ occurs precisely when $P$ is on the line passing through $A$ and $B$. The condition $PA - PB = 0$ occurs precisely when $P$ is on the perpendicular bisector of line segment $AB$. The condition $PB - PC = \pm 2$ occurs precisely when $P$ is on the line passing through $B$ and $C$. The condition $PB - PC = \pm 1$ occurs precisely when
P is on a specific hyperbola with foci at B and C (that the absolute value of the difference in the distances PB and PC is constant defines geometrically a hyperbola). The condition \( PB - PC = 0 \) occurs precisely when P is on the perpendicular bisector of line segment BC. We need both \( PA - PB \in \{-1, 0, 1\} \) and \( PB - PC \in \{-2, -1, 0, 1, 2\} \) to occur so that in the end we are interested in the number of intersection points between one of the two lines determined by \( PA - PB \in \{-1, 0, 1\} \) and one of the two lines and one hyperbola determined by \( PB - PC \in \{-2, -1, 0, 1, 2\} \). One counts the number of intersection points directly. For example, the line passing through A and B and the hyperbola defined above intersect in two points. It is of relevance when counting to note that the point \( Q = (0, 1/2) \) is not on the hyperbola (this can be seen by computing \( QB - QC \) and verifying that it is not \( \pm 1 \)). We deduce that there are [6] points P as in the problem.

**T5.** Using the identities

\[
\cos(A+B) = \cos(A) \cos(B) - \sin(A) \sin(B) \quad \text{and} \quad \cos(A-B) = \cos(A) \cos(B) + \sin(A) \sin(B)
\]

with \( A = n\theta \) and \( B = \theta \), one obtains

\[
(*) \quad \cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(\theta) \cos(n\theta).
\]

We take \( g_1(x) = x \) and \( g_2(x) = 2x^2 - 1 \) so that \( \cos(\theta) = g_1(\cos(\theta)) \) and \( \cos(2\theta) = g_2(\cos(\theta)) \). By setting \( g_{n+1}(x) = 2xg_n(x) - g_{n-1}(x) \) for \( n \geq 2 \), we get from \((*)\) that \( \cos(m\theta) = g_m(\cos(\theta)) \) for all positive integers \( m \). The formula \( g_{n+1}(x) = 2xg_n(x) - g_{n-1}(x) \) for \( n \geq 2 \) (together with the values given for \( g_1(x) \) and \( g_2(x) \)) allows one to compute \( g_n(x) = 32x^6 - 48x^4 + 18x^2 - 1 \). Letting \( u = \cos(\theta) \) with \( \theta = 10^\circ \), we deduce that

\[
\frac{1}{2} = \cos(60^\circ) = \cos(6\theta) = g_6(\cos(\theta)) = 32u^6 - 48u^4 + 18u^2 - 1.
\]

Thus, for \( u = \cos(\theta) \) with \( \theta = 10^\circ \), we have

\[
0 = 64u^6 - 96u^4 + 36u^2 - 3 = (2u)^6 - 6(2u)^4 + 9(2u)^2 - 3.
\]

It follows that \( 2\cos(10^\circ) = 2u \) is a root of \( \sqrt[3]{x^6 - 6x^4 + 9x^2 - 3} \) (The answer is unique.)

**T6.** Let A denote the die with one garnet face, and let R denote the garnet face. Let B denote the die with two garnet faces; denote them by \( S_1 \) and \( S_2 \). Let C denote the die with three garnet faces; denote them by \( T_1, T_2, \) and \( T_3 \). The possibilities for the roll of the three dice are:

<table>
<thead>
<tr>
<th>Two Faces Known</th>
<th>Third Face</th>
<th># of Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( R ) and an ( S_i )</td>
<td>( T_1, T_2, ) or ( T_3 )</td>
<td>6</td>
</tr>
<tr>
<td>2. ( R ) and an ( S_i )</td>
<td>one of 3 blue faces of die C</td>
<td>6</td>
</tr>
<tr>
<td>3. ( R ) and a ( T_j )</td>
<td>( S_1 ) or ( S_2 )</td>
<td>6</td>
</tr>
<tr>
<td>4. ( R ) and a ( T_j )</td>
<td>one of 4 blue faces of die B</td>
<td>12</td>
</tr>
<tr>
<td>5. an ( S_i ) and a ( T_j )</td>
<td>( R )</td>
<td>6</td>
</tr>
<tr>
<td>6. an ( S_i ) and a ( T_j )</td>
<td>one of 5 blue faces of die A</td>
<td>30</td>
</tr>
</tbody>
</table>
There are 66 cases above exactly 18 of which occur with the third face turning up garnet. The probability is 18/66 or (simplified) \( \frac{3}{11} \).

**T7.** If \( g(x) \) is a reciprocal polynomial of degree \( r \) (and non-zero constant term), then \( x^r g(1/x) = g(x) \). Suppose \( g(x) \) is a reciprocal polynomial of degree \( r \) dividing \( u(x) = x^{1234} - x^3 - x + 1 \) so that \( u(x) = g(x)h(x) \) for some polynomial \( h(x) \) of degree \( 1234 - r \). Then

\[
x^{1234} - x^{1233} - x^{1231} + 1 = x^{1234}u(1/x) = x^{1234}g(1/x)h(1/x)
= x^r g(1/x) x^{1234-r} h(1/x) = g(x) \left( x^{1234-r} h(1/x) \right).
\]

Thus, \( g(x) \) is a factor of \( w(x) = x^{1234} - x^{1233} - x^{1231} + 1 \). Since \( g(x) \) divides each of \( u(x) \) and \( w(x) \), we deduce that it divides

\[
(x^3 - x^2 - 1)u(x) = x^3 + (x^3 - x^2 - 1)(x^3 + x - 1)
= x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1 = (x - 1)^2(x^2 + 1)(x^2 + x + 1).
\]

It follows that \( g(x) \) must divide this last expression. In particular, the only possible factors of \( g(x) \) are \( x - 1, x^2 + 1, \) and \( x^2 + x + 1 \). Each of these is a factor of \( u(x) = x^{1234} - x^3 - x + 1 \) which can be verified as follows. First, \( u(1) = 0 \) and \( u(i) = 0 \) imply that \( x - 1 \) and \( x^2 + 1 \) are factors. Since \( x^2 + x + 1 \) is a factor of \( x^3 - 1 \) and \( x^3 - 1 \) is a factor of

\[
x(x^{3 \times 411} - 1) - (x^3 - 1) = x^{1234} - x^3 - x + 1,
\]

we get \( x^2 + x + 1 \) is a factor of \( u(x) \).

Next, we note that \( (x - 1)^2 \) does not divide \( u(x) \). Two ways to see this are:

(i) Use that if \( (x - 1)^2 \) is a factor of \( u(x) \), then \( u'(1) = 0 \) but \( u'(1) = 1230 \neq 0 \). (This is a Calculus approach; a non-Calculus approach follows.)

(ii) Use that if \( (x - 1)^2 \) is a factor of \( u(x) \), then \( u(x)^2 = (x - 1)^2 v(x) \) for some polynomial \( v(x) \) so that \( u(x + 1) = x^2 v(x + 1) \). This implies that the coefficient of \( x \) in \( u(x + 1) \) is 0. The Binomial Theorem implies, however, that the coefficient of \( x \) in \( u(x + 1) = (x + 1)^{1234} - (x + 1)^3 - (x + 1) + 1 \) is 1230.

Either way, we deduce that \( (x - 1)^2 \) is not a factor of \( u(x) \). It follows that \( g(x) \) divides \( (x-1)(x^2 + 1)(x^2 + x + 1) \). Since \( g(x) \) is reciprocal, the leading coefficient and constant term must be equal. It follows that \( x - 1 \) is not a factor of \( g(x) \). (Note that if \( (x - 1)^2 = x^2 - 2x + 1 \) could be a factor of \( g(x) \), the situation would be different.) The reciprocal factor of \( u(x) \) of largest degree is now seen to be \( (x^2 + 1)(x^2 + x + 1) = x^4 + x^3 + 2x^2 + x + 1 \).

**T8.** Observe that \( S < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \), so that the decimal representation of \( S \) in the problem is justified (i.e., there are no non-zero digits to the left of the decimal). We work with

\[
10^{24} S = d_1 d_2 d_3 \ldots d_{24} d_{25} d_{26} \ldots
\]
For \( n \geq 5 \), the value of \( 10^{24}/2^{n!} \) is less than or equal to \( 10^{24}/2^{5!+(n-5)} \) so that

\[
10^{24} \sum_{n=5}^{1997} \frac{1}{2^n!} \leq 10^{24} \frac{1}{2^{5!}} \sum_{n=5}^{1997} \frac{1}{2^{n-5}} < 10^{24} \sum_{n=0}^{\infty} \frac{1}{2^n} = 10^{24} \frac{1}{2^{119}} < \frac{10^{24}}{(10^{3/10})^{119}} < 1.
\]

On the other hand, for \( n \leq 4 \), the value of \( 10^{24}/2^{n!} \) is an integer, namely \( 2^{24-n!}5^{24} \). Observe that for \( n \leq 3 \), the number \( 10^{24}/2^{n!} = 2^{24-n!}5^{24} \) is divisible by \( 10^{24-3!} = 10^{18} \) so that the right 18 most digits of \( 10^{24}/2^{n!} \) are all 0. The sum of the three terms \( 10^{24}/2^{n!} \) for \( 1 \leq n \leq 3 \) in \( 10^{24}S \) therefore contribute nothing to the first 18 digits to the left of the decimal in \( 10^{24}S \). It remains to consider \( 10^{24}/2^{11} \) with \( n = 4 \). We have \( 10^{24}/2^{4!} = 5^{24} \). We compute the right most five digits of \( 5^{24} \). This can be done easily with a calculator. For example, \( 5^8 = 390625 \). We are only interested in the last 5 right most digits, so to calculate the last 5 digits of \( 5^{12} \) we can simply compute the value of \( 90625 \times 625 \) which ends with 40625. If the calculator can calculate \( 40625 \times 40625 \) exactly, then we can get the last 5 digits of \( 10^{24} \) by computing the last 5 digits in this product. Alternatively, one can multiply by 625 three times (each time dropping all but the last 5 digits). This gives that the last 5 digits of \( 5^{24} \) are 90625. The information above now implies that \( d_{20} = 9, d_{21} = 0, d_{22} = 6, d_{23} = 2, \) and \( d_{24} = 5 \). Hence, the sum in the problem is \( 9 + 0 + 6 + 2 + 5 = 22 \).

**T9.** Let \( B' = (13, 4) \), and let \( P \) denote the intersection of the \( x \)-axis with the line passing through \( A \) and \( B' \). We claim that with \( P \) so chosen, \( AP - BP \) gives the desired maximum.

To see this observe that if \( Q \) is any point on the \( x \)-axis, then \( AQ - BQ = AQ - B'Q \). In particular, \( AP - BP = AB' = \sqrt{12^2 + 5^2} = 13 \). On the other hand, if \( Q \neq P \), then since the sum of the lengths of two sides of a triangle is greater than the third, we obtain \( AB' + B'Q > AQ \) so that \( AQ - BQ = AQ - B'Q < AB' = 13 \). Hence, the answer is \( 13 \).

**T10.** Let \( \alpha = 628318530717958647692528/10^{23} \), and observe that \( \alpha \approx 2\pi \). Since \( \alpha > 1 \), we cannot use the estimate given in the problem for \( \cos x \) directly. However, since \( \cos \alpha = \cos(2\pi - \alpha) \), we can use the estimate by setting \( x = 2\pi - \alpha \). Since \( 2\pi \times 10^{23} = 628318530717958647692528.6766559\ldots \), we obtain that

\[
\cos \alpha = \cos(2\pi - \alpha) = 1 - \frac{1}{2} \left( \frac{0.6766559\ldots}{10^{23}} \right)^2 + E \left( \frac{0.6766559\ldots}{10^{23}} \right).
\]

From the information given in the problem, \( E(x) < 1/10^{92} \). Since \( \frac{1}{2}(0.6766559\ldots)^2 = 0.22893\ldots \), it follows that \( \cos \alpha = 1 - u + v \) where \( u < 1 \), the decimal expansion of \( u \) consists of 46 zeroes to the right of the decimal followed by the digits 22893, and the decimal expansion of \( v \) consists of at least 92 zeroes to the right of the decimal. Hence, \( \cos \alpha = 0.999\ldots 9977106\ldots \), where 46 nines occur between the decimal and the first digit seven. The answer is \( 1 \).