Solutions to 1994 TEAM PROBLEMS

T1. The sum of the grades in the class is $20 \times 72.45 = 1449$. This is also the sum of the 18 remaining students' grades. Thus, their average is $1449/18 = 80.5$.

T2. The Rational Root Test implies that the only "possible" rational roots are of the form a/b where a divides 1155 and b divides 1. Since $1155 = 3 \times 5 \times 7 \times 11$, this means that there are 32 possible rational roots ($\pm 1, \pm 3, \pm 5, \pm 7, \pm 11, \pm 15, \pm 21, \dots$). Observe that all 32 of these numbers are odd. But if x is an odd number, then $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155$ is odd and, therefore, not 0. It follows that there are no rational roots to the given equation. The answer is 0.

T3. Use the equation $2^{\log_2 x} = x$ to obtain

$$f(x) = x^{g(h(x))} = x^{1/\log_2 x} = \left(2^{\log_2 x}\right)^{1/\log_2 x} = 2.$$

Thus, $f(x)$ is 2 for all values of x, and the answer is 2.

T4. The answer is $A = 999$. The set $S = 0, 1, 2, \dots, 997$ satisfies (i) since it has $998 = A - 1$ elements and since $1 \leq a + b \leq 996 + 997 = 1993$ for any two elements a and b from this set S. To see that (ii) holds, suppose we have a set T consisting of $A = 999$ consecutive integers. We can write T in the form $T = \{u, u+1, u+2, \dots, u+998\}$ for some integer u. Observe that $u + (u + j) = 2u + j$ for $1 \leq j \leq 998$ so each of the numbers $2u+1, 2u+2, 2u+3, \dots, 2u+998$ can be written in the form $a + b$ for some distinct a and b from T. Also, $(u + j) + (u + 998) = 2u + 998 + j$ for $1 \leq j \leq 997$ so $2u + 999, 2u + 1000, 2u + 1001, \dots, 2u + 1995$ can also be written in the form $a + b$ for some distinct a and b from T. In other words, there are 1995 consecutive integers from $2u + 1$ to $2u + 1995$ each of which can be written in the form $a + b$ for some distinct a and b from T. At least one of these must be divisible by 1994, so (ii) holds.

T5. An important observation here is that (a, b) is visible from the origin if and only if the greatest common divisor of a and b is 1. Take $b = 2 \times 3 \times 5 \times 7 = 210$. Then each of $2, 3, 4, \dots, 8$ has a common factor > 1 with b, so b satisfies the condition in the problem. (Note: 210 is the only answer to this problem.)

T6. Descartes' Rule of Signs gives that there is exactly one positive real root and exactly one negative real root of the given polynomial. Observe that $\pm\sqrt{2}$ are roots of the given polynomial. Hence, the only real roots to the given polynomial are $\pm\sqrt{2}$, and the answer is $(\sqrt{2})^2 + (-\sqrt{2})^2 = 2 + 2 = 4$.

T7. Observe that

$$-\sqrt{2501} + \sqrt{2502} = \frac{1}{\sqrt{2501} + \sqrt{2502}} < \frac{1}{50 + 50} = \frac{1}{100}.$$

Similarly, each of $-\sqrt{2503}+\sqrt{2504}, -\sqrt{2505}+\sqrt{2506}, \ldots, -\sqrt{2999}+\sqrt{3000}$ is $< 1/100$. We get that

$$\begin{aligned}\sqrt{2500}-\sqrt{2501}+&\sqrt{2502}-\sqrt{2503}+\cdots-\sqrt{2999}+\sqrt{3000}\\ &< \sqrt{2500}+\frac{1}{100}+\cdots+\frac{1}{100}=50+\frac{250}{100}=52.5.\end{aligned}$$

We claim the answer is 52. We need only show now that

$$\sqrt{2500}-\sqrt{2501}+\sqrt{2502}-\sqrt{2503}+\cdots-\sqrt{2999}+\sqrt{3000}>52.$$

In other words, we want

$$\sqrt{3000}>52-\sqrt{2500}+\sqrt{2501}-\sqrt{2502}+\sqrt{2503}-\cdots+\sqrt{2999}.$$

An argument as above gives

$$52-\sqrt{2500}+\sqrt{2501}-\cdots+\sqrt{2999}<52+250\left(\frac{1}{100}\right)=54.5,$$

so we need only show that $\sqrt{3000}>54.5$ (this is correct – do you understand why?) which follows since $3000>2970.25=54.5^2$.

T8. The number a_2 is a power of 2. Observe that the last digits of $2, 2^2, 2^3, 2^4, \ldots$ are $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. In other words, $2^{4k+1}, 2^{4k+2}, 2^{4k+3}, 2^{4k+4}$ end with $2, 4, 8, 6$, respectively, for each non-negative integer k. Since $a_2 = 2^{a_3}$, we want to know in which of the forms $4k+1, 4k+2, 4k+3, 4k+4$ we can write a_3. Since a_3 is a power of 3, we consider successive powers of 3. Note that $3, 3^2, 3^3, 3^4, \ldots$ are in the form $4k+3, 4k+1, 4k+3, 4k+1, \ldots$ (for different k's). In other words, if 3 is raised to an odd power, then the result is of the form $4k+3$; and if 3 is raised to an even power, then the result is of the form $4k+1$. Since $a_3 = 3^{a_4}$ and $a_4 = 4^{a_5}$, a_3 is 3 raised to an even power and, hence, of the form $4k+1$ for some positive integer k. We get that the final digit of $a_2 = 2^{a_3}$ is 2.

T9. If two of the points are (x_1, y_1) and (x_2, y_2), then the slope of the line through the two points is $(y_2-y_1)/(x_2-x_1)$. Also,

$$y_1 = 2x_1^3 - 4x_1 + 2,\ y_2 = 2x_2^3 - 4x_2 + 2,$$

$$y_1 = x_1^3 + 2x_1 - 1, \text{ and } y_2 = x_2^3 + 2x_2 - 1.$$

Taking the difference of the first two of these, we get that

$$y_2 - y_1 = 2(x_2^3 - x_1^3) - 4(x_2 - x_1) = (x_2 - x_1)\left(2(x_2^2 + x_2x_1 + x_1^2) - 4\right).$$

Hence, the slope of the line through the two points is

$$\frac{y_2-y_1}{x_2-x_1} = 2(x_2^2 + x_2x_1 + x_1^2) - 4.$$

Taking another difference of equations above gives

$$y_2 - y_1 = x_2^3 - x_1^3 + 2(x_2 - x_1) = (x_2 - x_1)\left(x_2^2 + x_2x_1 + x_1^2 + 2\right).$$

Hence, the slope of the line through the two points is also

$$\frac{y_2 - y_1}{x_2 - x_1} = x_2^2 + x_2x_1 + x_1^2 + 2.$$

This means that

$$2(x_2^2 + x_2x_1 + x_1^2) - 4 = x_2^2 + x_2x_1 + x_1^2 + 2$$

so that $x_2^2 + x_2x_1 + x_1^2 = 6$. Substituting this into either of the two formulas for the slope above, we get that the slope is 8. (Comment: Since the answer does not depend on which two points we consider, we can conclude that the three points of intersection all lie on a straight line.)

T10. We use the notation PQ to represent the distance between two points P and Q. Let O represent the origin. Let E be a point on $\overleftrightarrow{OA}$ such that $\overleftrightarrow{CE}$ is perpendicular to $\overleftrightarrow{OA}$. Observe that $\angle CEA = \angle DBA = 90°$. One easily gets that ΔCEA is similar to ΔDBA. Hence, $AB/AD = AE/AC$ so that $AB \times AC = AE \times AD$. The given information implies that $AB \times AC = 1$ and $AD = 2$. Therefore, $AE = 1/2$ which implies $OE = 1/2$. It follows that the $x-$coordinate of C is $1/2$. (Note that the measurements in the drawing are not correct. Also, 17 has nothing to do with this answer.)