SOLUTIONS TO TEAM PROBLEMS 02/99

ANSWERS:  T1: 11/32  T5: $(\pi + 8)/4$
T2: 43046721  T6: $3/\sqrt{10}$
T3: $30^\circ$ or $\pi/6$ radians  T7: 89
T4: 5547  T8: 5

T1. There are $2^6 = 64$ possible outcomes for the 6 coin flips. The number of these with heads landing face-up more often than tails is the number of times heads lands face-up on four flips of the coin plus the number of times heads lands face-up on 5 flips plus the number of times heads lands face-up on all 6 flips. This is the same as

$$\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 15 + 6 + 1 = 22.$$

Therefore, the probability of heads being face-up more often than tails is $22/64 = 11/32$.

T2. The binomial theorem asserts that

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

Take $n = 16$, $x = 2$, and $y = 1$ to get

$$\sum_{j=0}^{16} 2^j \binom{16}{j} = (2 + 1)^{16} = 43046721.$$  

T3. The measure of $\angle C$ is one-half of the measure of $\angle AOB$ where 0 denotes the center of the circle. On the other hand, each side of $\Delta AOB$ has length one, so $\Delta AOB$ is an equilateral triangle. Thus, $\angle AOB = 60^\circ$ so that $\angle C = 30^\circ$.

Another solution is as follows. The measure of $\angle C$ does not change if the location of $C$ is moved to another point on the circle on the same side of line $AB$. We move $C$ so that line segment $AC$ passes through the center of the circle. It follows that $AC = 2$ (the diameter of the circle). Also, $\angle ABC = 90^\circ$. Since we are given that $AB = 1$, we deduce that $\sin \angle C = 1/2$ and, hence, $\angle C = 30^\circ$.

T4. Observe that if $q$ is the quotient and $r$ is the remainder when $10^{100}$ is divided by 1999, then $10^{100} = 1999q + r$ so that

$$10^{100} \times \frac{1}{1999} = q + \frac{r}{1999}.$$
It follows that \( r/1999 = 0.d_{101}d_{102}d_{103}d_{104} \ldots \). We determine \( r \) by computing \( 10^{100} \) modulo 1999. With the help of a calculator, we obtain

\[
10^4 \equiv 5 \pmod{1999} \implies 10^{20} \equiv 5^5 \equiv 1126 \pmod{1999}
\]

\[
\implies 10^{40} \equiv 1126^2 \equiv 510 \pmod{1999} \implies 10^{80} \equiv 510^2 \equiv 230 \pmod{1999}
\]

\[
\implies 10^{100} \equiv 10^{20} \times 10^{80} \equiv 1126 \times 230 \equiv 1109 \pmod{1999}.
\]

Thus, \( r = 1109 \). Since \( 1109/1999 = .554777 \ldots \), we deduce \( d_{101}d_{102}d_{103}d_{104} = 5547 \).

**T5.** The \( x \)-coordinates where the graphs intersect satisfy \( 4x^3 - \pi x^2 + 3x - 1 = 8x^2 - 5 \) so that \( 4x^3 - (\pi + 8)x^2 + 3x + 4 = 0 \). The sum of the roots of this equation is \( (\pi + 8)/4 \). We deduce that \( x_1 + x_2 + x_3 = (\pi + 8)/4 \).

**T6.** Let \( \theta \) be such that \( \cos \theta = 1/\sqrt{10} \) and \( \sin \theta = 3/\sqrt{10} \) (you should justify that such a \( \theta \) exists). We rewrite the function \( f(x) \) as follows:

\[
f(x) = \cos x + 3 \sin x = \sqrt{10} \left( \cos \theta \cos x + \sin \theta \sin x \right) = \sqrt{10} \cos (\theta - x).
\]

Taking \( t = \theta \), we see that

\[
f(t) = \sqrt{10} \geq \sqrt{10} \cos (\theta - x)
\]

for every \( x \). Hence, \( t = \theta \) gives the value of \( t \) in the statement of the problem. Therefore, the answer is \( \sin t = \sin \theta = 3/\sqrt{10} \). (Note that there is more than one \( t \) as in the problem, but \( \sin t \) is uniquely determined.)

**T7.** Let \( D_n \) denote the number of ways of covering a \( 2 \times n \) board with dominoes. One checks directly that \( D_1 = 1 \) and \( D_2 = 2 \). If \( n \geq 3 \), then every covering of a \( 2 \times n \) board must either have its last column covered by a \( 2 \times 1 \) domino or its last two columns covered by two \( 1 \times 2 \) dominoes. It follows that each covering of a \( 2 \times n \) board can be constructed from either a covering of a \( 2 \times (n-1) \) board followed by a \( 2 \times 1 \) domino or a covering of a \( 2 \times (n-2) \) board followed by two \( 1 \times 2 \) dominoes. In other words, \( D_n = D_{n-1} + D_{n-2} \) for each \( n \geq 3 \). A direct computation now gives \( D_3 = D_2 + D_1 = 2 + 1 = 3 \), \( D_4 = D_3 + D_2 = 3 + 2 = 5 \), \( D_5 = D_4 + D_3 = 5 + 3 = 8 \), \ldots, \( D_{10} = 55 + 34 = 89 \).

**T8.** There are a variety of different polynomials \( g(x) \) as in the problem. We begin by explaining one way to obtain such a \( g(x) \). Let \( g(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \), and let \( f(x) = (x^2 - 4x + 5)g(x) \). We consider \( a_n = 1 \) and integers \( a_{n-1}, a_{n-2}, \ldots \) successively as small as possible so that the the leading coefficients in \( f(x) \) are nonnegative. Observe that

\[
(x^2 - 4x + 5)g(x) = (x^2 - 4x + 5)(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots)
\]

\[
= x^{n+2} + (a_{n-1} - 4)x^{n+1} + (a_{n-2} - 4a_{n-1} + 5)x^n
\]

\[+(a_{n-3} - 4a_{n-2} + 5a_{n-1})x^{n-1} + \cdots.
\]
We take $a_{n-1} = 4$, $a_{n-2} = 11$, and then $a_{n-3} = 24$. We continue in this manner; but to help with the computations, note that the coefficient of $x^{n-j}$ is $a_{n-j-2} - 4a_{n-j-1} + 5a_{n-j}$ for $j \geq 1$ so that $a_{n-j-2} \geq 4a_{n-j-1} - 5a_{n-j}$. Hence, we take

$$a_{n-4} = 4 \times 24 - 5 \times 11 = 41 \quad \text{and} \quad a_{n-5} = 4 \times 41 - 5 \times 24 = 44.$$ 

Since $a_{n-6} \geq 4 \times 44 - 5 \times 41 = -29$, a negative number $a_{n-6}$ will make the coefficient of $x^{n-4}$ in $f(x)$ nonnegative. This suggests that perhaps $a_{n-6}$ is unnecessary (it can be 0), so we attempt to take $n = 5$ and $g(x) = x^5 + 4x^4 + 11x^3 + 24x^2 + 41x + 44$. One obtains that $(x^2 - 4x + 5)g(x) = x^7 + 29x + 220$ in this case, so that $(x^2 - 4x + 5)g(x)$ does indeed have nonnegative coefficients.

It remains to show that $g(x)$ cannot be replaced by a polynomial of degree < 5. It suffices to show that $f(x) = (x^2 - 4x + 5)g(x)$ cannot be of degree < 7. Recall that deg $f \geq 2$ since $g(x)$ is not identically 0. Suppose $f(x)$ is a polynomial of degree $\geq 2$ and $\leq 6$ with nonnegative coefficients. Note that the roots of $x^2 - 4x + 5$ are $2 \pm i$. We consider the root $2 + i = \sqrt{5}e^{i\theta}$ where $\theta = \tan^{-1}(1/2) \in (0, 0.4635)$. It follows that

$$(2 + i)^k = \sqrt{5}^k e^{ik\theta} = \sqrt{5}^k (\cos(k\theta) + i\sin(k\theta)).$$

Observe that for $1 \leq k \leq 6$, we have $0 < k\theta < 6 \times 0.4635 = 2.7819 < \pi$. Hence, the imaginary part of $(2 + i)^k$ is positive for $1 \leq k \leq 6$. If $f(x) = \sum_{k=0}^n c_k x^k$ with $c_n \neq 0$, $2 \leq n \leq 6$, and $c_k \geq 0$ for each $k$, then it follows that the imaginary part of $f(2 + i) = \sum_{k=0}^n c_k (2 + i)^k$ is > 0. This implies that $f(x)$ cannot have $2 + i$ as a root and, therefore, cannot have $x^2 - 4x + 5$ as a factor. Thus, if $f(x) = (x^2 - 4x + 5)g(x)$ with $g(x)$ as in the problem, then deg $f \geq 7$. 