SOLUTIONS TO TEAM PROBLEMS 02/99

ANSWERS: T1:	$11/32$	T5:	$(\pi+8)/4$
T2:	43046721	T6:	$3/\sqrt{10}$
T3:	$30°$ or $\pi/6$ radians	T7:	89
T4:	5547	T8:	5

T1. There are $2^6 = 64$ possible outcomes for the 6 coin flips. The number of these with heads landing face-up more often than tails is the number of times heads lands face-up on four flips of the coin plus the number of times heads lands face-up on 5 flips plus the number of times heads lands face-up on all 6 flips. This is the same as

$$\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 15 + 6 + 1 = 22.$$

Therefore, the probability of heads being face-up more often than tails is $22/64 = 11/32$.

T2. The binomial theorem asserts that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

Take $n = 16$, $x = 2$, and $y = 1$ to get

$$\sum_{j=0}^{16} 2^j \binom{16}{j} = (2+1)^{16} = 43046721.$$

T3. The measure of $\angle C$ is one-half of the measure of $\angle AOB$ where 0 denotes the center of the circle. On the other hand, each side of ΔAOB has length one, so ΔAOB is an equilateral triangle. Thus, $\angle AOB = 60°$ so that $\angle C = 30°$.

Another solution is as follows. The measure of $\angle C$ does not change if the location of C is moved to another point on the circle on the same side of line $\overleftrightarrow{AB}$. We move C so that line segment $\overline{AC}$ passes through the center of the circle. It follows that $AC = 2$ (the diameter of the circle). Also, $\angle ABC = 90°$. Since we are given that $AB = 1$, we deduce that $\sin\angle C = 1/2$ and, hence, $\angle C = 30°$.

T4. Observe that if q is the quotient and r is the remainder when 10^{100} is divided by 1999, then $10^{100} = 1999q + r$ so that

$$10^{100} \times \frac{1}{1999} = q + \frac{r}{1999}.$$

It follows that $r/1999 = 0.d_{101}d_{102}d_{103}d_{104}\ldots$. We determine r by computing 10^{100} modulo 1999. With the help of a calculator, we obtain

$$10^4 \equiv 5 \pmod{1999} \implies 10^{20} \equiv 5^5 \equiv 1126 \pmod{1999}$$

$$\implies 10^{40} \equiv 1126^2 \equiv 510 \pmod{1999} \implies 10^{80} \equiv 510^2 \equiv 230 \pmod{1999}$$

$$\implies 10^{100} \equiv 10^{20} \times 10^{80} \equiv 1126 \times 230 \equiv 1109 \pmod{1999}.$$

Thus, $r = 1109$. Since $1109/1999 = .554777\ldots$, we deduce $d_{101}d_{102}d_{103}d_{104} = 5547$.

T5. The x-coordinates where the graphs intersect satisfy $4x^3 - \pi x^2 + 3x - 1 = 8x^2 - 5$ so that $4x^3 - (\pi + 8)x^2 + 3x + 4 = 0$. The sum of the roots of this equation is $(\pi + 8)/4$. We deduce that $x_1 + x_2 + x_3 = (\pi + 8)/4$.

T6. Let θ be such that $\cos\theta = 1/\sqrt{10}$ and $\sin\theta = 3/\sqrt{10}$ (you should justify that such a θ exists). We rewrite the function $f(x)$ as follows:

$$f(x) = \cos x + 3\sin x = \sqrt{10}\left(\cos\theta\cos x + \sin\theta\sin x\right) = \sqrt{10}\cos(\theta - x).$$

Taking $t = \theta$, we see that

$$f(t) = \sqrt{10} \geq \sqrt{10}\cos(\theta - x)$$

for every x. Hence, $t = \theta$ gives the value of t in the statement of the problem. Therefore, the answer is $\sin t = \sin\theta = 3/\sqrt{10}$. (Note that there is more than one t as in the problem, but $\sin t$ is uniquely determined.)

T7. Let D_n denote the number of ways of covering a $2 \times n$ board with dominoes. One checks directly that $D_1 = 1$ and $D_2 = 2$. If $n \geq 3$, then every covering of a $2 \times n$ board must either have its last column covered by a 2×1 domino or its last two columns covered by two 1×2 dominoes. It follows that each covering of a $2 \times n$ board can be constructed from either a covering of a $2 \times (n-1)$ board followed by a 2×1 domino or a covering of a $2 \times (n-2)$ board followed by two 1×2 dominoes. In other words, $D_n = D_{n-1} + D_{n-2}$ for each $n \geq 3$. A direct computation now gives $D_3 = D_2 + D_1 = 2 + 1 = 3$, $D_4 = D_3 + D_2 = 3 + 2 = 5$, $D_5 = D_4 + D_3 = 5 + 3 = 8$, $\ldots$, $D_{10} = 55 + 34 = 89$.

T8. There are a variety of different polynomials $g(x)$ as in the problem. We begin by explaining one way to obtain such a $g(x)$. Let $g(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots$, and let $f(x) = (x^2 - 4x + 5)g(x)$. We consider $a_n = 1$ and integers $a_{n-1}, a_{n-2}, \ldots$ successively as small as possible so that the the leading coefficients in $f(x)$ are nonnegative. Observe that

$$\begin{aligned}(x^2 - 4x + 5)g(x) &= (x^2 - 4x + 5)(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots) \\ &= x^{n+2} + (a_{n-1} - 4)x^{n+1} + (a_{n-2} - 4a_{n-1} + 5)x^n \\ &\quad + (a_{n-3} - 4a_{n-2} + 5a_{n-1})x^{n-1} + \cdots .\end{aligned}$$

We take $a_{n-1} = 4$, $a_{n-2} = 11$, and then $a_{n-3} = 24$. We continue in this manner; but to help with the computations, note that the coefficient of x^{n-j} is $a_{n-j-2} - 4a_{n-j-1} + 5a_{n-j}$ for $j \geq 1$ so that $a_{n-j-2} \geq 4a_{n-j-1} - 5a_{n-j}$. Hence, we take

$$a_{n-4} = 4 \times 24 - 5 \times 11 = 41 \quad \text{and} \quad a_{n-5} = 4 \times 41 - 5 \times 24 = 44.$$

Since $a_{n-6} \geq 4 \times 44 - 5 \times 41 = -29$, a negative number a_{n-6} will make the coefficient of x^{n-4} in $f(x)$ nonnegative. This suggests that perhaps a_{n-6} is unnecessary (it can be 0), so we attempt to take $n = 5$ and $g(x) = x^5 + 4x^4 + 11x^3 + 24x^2 + 41x + 44$. One obtains that $(x^2 - 4x + 5)g(x) = x^7 + 29x + 220$ in this case, so that $(x^2 - 4x + 5)g(x)$ does indeed have nonnegative coefficients.

It remains to show that $g(x)$ cannot be replaced by a polynomial of degree < 5. It suffices to show that $f(x) = (x^2 - 4x + 5)g(x)$ cannot be of degree < 7. Recall that $\deg f \geq 2$ since $g(x)$ is not identically 0. Suppose $f(x)$ is a polynomial of degree ≥ 2 and ≤ 6 with nonnegative coefficients. Note that the roots of $x^2 - 4x + 5$ are $2 \pm \mathrm{i}$. We consider the root $2 + \mathrm{i} = \sqrt{5}\mathrm{e}^{\mathrm{i}\theta}$ where $\theta = \tan^{-1}(1/2) \in (0, 0.46365)$. It follows that

$$(2 + \mathrm{i})^k = \sqrt{5}^k \mathrm{e}^{\mathrm{i}k\theta} = \sqrt{5}^k (\cos(k\theta) + \mathrm{i}\sin(k\theta)).$$

Observe that for $1 \leq k \leq 6$, we have $0 < k\theta < 6 \times 0.46365 = 2.7819 < \pi$. Hence, the imaginary part of $(2 + \mathrm{i})^k$ is positive for $1 \leq k \leq 6$. If $f(x) = \sum_{k=0}^{n} c_k x^k$ with $c_n \neq 0$, $2 \leq n \leq 6$, and $c_k \geq 0$ for each k, then it follows that the imaginary part of $f(2 + \mathrm{i}) = \sum_{k=0}^{n} c_k (2 + \mathrm{i})^k$ is > 0. This implies that $f(x)$ cannot have $2 + \mathrm{i}$ as a root and, therefore, cannot have $x^2 - 4x + 5$ as a factor. Thus, if $f(x) = (x^2 - 4x + 5)g(x)$ with $g(x)$ as in the problem, then $\deg f \geq 7$.