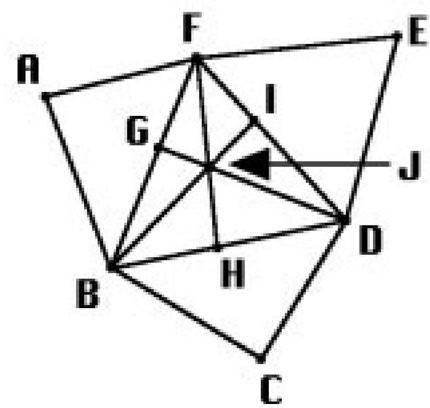
TEAM PROBLEMS January, 1998

(Calculators are permitted on this competition.)

T1. Let $A = (-1, 7)$, $B = (11, 3)$, $C = (-2, 0)$, and $D = (14, 0)$. There are three points $P = (x_0, 0)$ on the x-axis such that $\angle APC + \angle BPD = 90°$. What are the three possible values of x_0? Simplify your answers.

T2. Given 10 points (labelled $A, B, \ldots, J$) and 18 edges (line segments) joining them as shown, a path is made by beginning with one of the points, traversing an edge to another point, traversing an edge again to a point, and so on. Consider only paths where each edge is traversed at most once (each point may occur more than once). In such a path, what is the maximum number of edges that can be traversed?



T3. There are positive integers a and b for which

$$\sum_{k=2}^{1000000} \frac{1}{\sqrt{k} + \sqrt{k+1}} = \sqrt{a} - \sqrt{b}.$$

What are a and b?

T4. A number is chosen at random from the set $\{1, 2, 3, \ldots, 1998\}$. What is the probability that it has no prime factors in common with 10! (i.e., 10 factorial)?

T5. An ellipse is symmetric about the origin and passes through the points $(1, 0)$, $(0, 1)$, and $(1, 1)$. Determine the point (x, y) on the ellipse with the largest possible value of y. Give explicit values for both x and y and simplify your answers.

T6. Let $f(x)$ be the polynomial one obtains by expanding the product

$$(x-1)(x-2)(x-3)\cdots(x-1000).$$

Let k be the coefficient of x^{125} in $f(x)$. There is a non-negative integer n such that 2^n divides k and 2^{n+1} does not divide k. What is the value of n?

T7. Find the sum of the four smallest distinct primes which divide the number

$$15^{\left(15^{15}\right)} + 15.$$

T8. Calculate and simplify $\displaystyle\sum_{j=1}^{99999} \cos^4\left(\frac{2\pi j}{99999}\right)$.