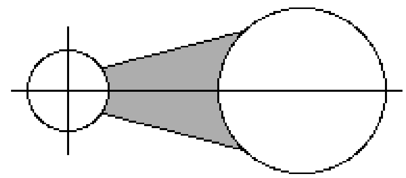
TEAM PROBLEMS* February, 1997

T1. How many divisors does the number 30! (thirty factorial) have?

T2. What is the area of the region between the two circles $x^2 + y^2 = 16$ and $(x - 16)^2 + y^2 = 100$ that lies under the line joining $(2\sqrt{3}, 2)$ to $(11, 5\sqrt{3})$ and above the line joining $(2\sqrt{3}, -2)$ to $(11, -5\sqrt{3})$? (See the shaded region in the picture, not drawn to scale.) Express your answer in the form $a + b\sqrt{3} + c\pi$ where a, b, and c are integers.



T3. Beginning with the origin $(0, 0)$ in the xy-plane, a path is made by taking steps where each step consists of either (i) adding 1 to both coordinates or (ii) adding 1 to the x-coordinate and subtracting 1 from the y-coordinate. For example, an acceptable path would be one going from $(0, 0)$ to $(1, 1)$ and then to $(2, 2)$; another acceptable path would be one going from $(0, 0)$ to $(1, -1)$ to $(2, 0)$ and then to $(3, 1)$. Each point occurring on such a path, we call a *thisforlackofabettername* point. For example, each of the points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(1, -1)$, $(2, 0)$, and $(3, 1)$ mentioned above is a thisforlackofabettername point. On the other hand, $(1, 0)$ and $(1, 3)$ are not. How many thisforlackofabettername points (x, y) are there satisfying $0 \le x \le 10$ and $0 \le y \le 10$?

T4. Consider the points $A = (0, 1)$, $B = (0, 0)$, and $C = (2, 0)$ in the Cartesian plane. How many points P in the plane have the property that $PA - PB$, $PA - PC$, and $PB - PC$ are all integers? Here PA represents the distance from the point P to the point A and similarly for PB and PC. Do not assume that the coordinates of P are necessarily integers.

T5. The number $2\cos(10°)$ is a root of a polynomial $f(x)$ satisfying: (i) the degree of $f(x)$ is 6, (ii) the coefficient of x^6 is 1, and (iii) the coefficients of $f(x)$ are integers. What is $f(x)$?

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*Calculators are allowed to be used during this competition. The problems are written accordingly. Also, note that solutions to the problems will be available through the World Wide Web at the URL **http://www.math.sc.edu/~filaseta/contests.html**.

T6. Three dice (cubes) are thrown simultaneously, one with 1 garnet face and 5 black faces, one with 2 garnet faces and 4 black faces, and one with 3 garnet faces and 3 black faces. Two of the dice turn up garnet (i.e., with a garnet face on top). What is the probability that the third die also turns up garnet? (Note that you know that two of the dice turn up garnet but you don't know which ones they are.)

T7. A *reciprocal* polynomial is a polynomial $f(x) = \sum_{j=0}^{n} a_j x^j$ satisfying

$$a_0 = a_n, \; a_1 = a_{n-1}, \; a_2 = a_{n-2}, \; \ldots, \; a_n = a_0.$$

For example, $x^5 - 2x^4 - 2x + 1$ and $3x^2 - 4x + 3$ are reciprocal polynomials. Consider all the reciprocal polynomials with integer coefficients that are factors of $x^{1234} - x^3 - x + 1$. Find the factor which has the largest degree. (Hint: Your answer should have degree 4.)

T8. Let $S = \displaystyle\sum_{n=1}^{1997} \frac{1}{2^{n!}}$. Suppose that the decimal expansion of S is given by $S = 0.d_1 d_2 d_3 \ldots$ (so that each of the d_j's represents a digit). What is the value of the sum $d_{20} + d_{21} + d_{22} + d_{23} + d_{24}$?

T9. Let $A = (1,9)$ and $B = (13,-4)$ be points in the xy-plane. For each point P on the x-axis, consider the distance from A to P (denoted AP) and the distance from B to P (denoted BP). What is the maximum value of the difference $AP - BP$?

T10. Given that

$$\pi = 3.14159265358979323846264338327950288419716939937 5\ldots$$

is accurate to the digits shown and that

$$\cos x = 1 - \frac{x^2}{2} + E(x) \quad \text{where } |E(x)| \le \frac{x^4}{24} \text{ if } |x| \le 1,$$

calculate the sum of the 49th and 50th digits after the decimal in the decimal expansion of

$$\cos\left(\frac{628318530717958647692528}{10^{23}}\right).$$

Throughout this problem, the angles are to be considered to be measured in radians. Note that the numerator in the last expression above contains 24 digits.