TEAM PROBLEMS February, 1996

T1. The number which is four more than the square of 625 has exactly two prime factors. Determine what they are.

T2. A circle of radius 1 rolls (without sliding) along the x-axis so that its center is of the form $(t, 1)$ with t increasing. A certain point P touches the x-axis at the origin as the circle rolls. As the circle rolls further, the point P touches the point $(x, 1/2)$. Given that $x \leq 3$, what is the value of x?

T3. Let α_1, α_2, and α_3 denote the roots of $x^3 - x + 1 = 0$. Calculate the value of

$$\alpha_1^4 + \alpha_2^4 + \alpha_3^4.$$

Simplify your answer. It should be an integer.

T4. An experiment consists of choosing two points P and Q at random from the circumference of a given circle of radius one. The average value of the distance from P to Q is $4/\pi$. What is the average value of the square of the distance from P to Q?

T5. A pentagon (not necessarily a regular pentagon) has vertices A, B, C, D, and E. Given that

the midpoint of edge $\overline{AB}$ is $(1, 2)$
the midpoint of edge $\overline{BC}$ is $(3, 0)$
the midpoint of edge $\overline{CD}$ is $(8, 1)$
the midpoint of edge $\overline{DE}$ is $(5, 4)$
the midpoint of edge $\overline{EA}$ is $(3, 4)$,

find the coordinates of A.

T6. Suppose

$$(x^2 - x + 1)^3(x^3 + 2x^2 + 2x + 1)^5 = a_{21}x^{21} + a_{20}x^{20} + \cdots + a_1 x + a_0.$$

What is the value of $a_1 + a_2 + \cdots + a_{10}$?

T7. For $k \geq 1$, define a_k by $a_k = 2^k + 3$. Thus, $a_1 = 5, a_2 = 7, a_3 = 11, \ldots$. It is not hard to see that every a_k is not divisible by 2 and every a_k is not divisible by 3. Find the next two primes p having the property that every a_k is not divisible by p.

T8. Two honest people, named Bert and Ernie, each make you an offer as indicated below. Determine who is making the better offer and determine an appropriate statment you might make. The statement should clarify why the particular person you claim is making the better offer is the person who is making the better offer.

Bert's Offer: Make a statement. If it is true, then I will give you exactly ten dollars. If it is false, then I will give you either less than ten dollars or more than ten dollars but not exactly ten dollars.

Ernie's Offer: Make a statement. If it is true, then I will give you more than ten dollars. If it is false, then I will give you more than ten dollars. (That's right, either way I will give you more than ten dollars.)

T9. A chessboard contains 64 one-by-one squares, but it also contains some two-by-two squares, some three-by-three squares, and so on. In particular, there is one eight-by-eight square on a chessboard. How many total squares are there on a chessboard? Simplify your answer.

T10. How many total rectangles are there on a chessboard (see Problem T9)? (To clarify, a "rectangle" must have positive area. In other words, we do not allow here for one of its dimensions to be zero.) Simplify your answer.