1994 TEAM PROBLEMS

T1. The average grade of 20 students in a class on a test is 72.45 points. Two of the 20 students received 0 points on the test (because they didn't show up for the test). Calculate the average grade of the remaining 18 students.

T2. Calculate the number of rational roots of the equation

$$x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0.$$

T3. Let $g(x) = 1/x$ and $h(x) = \log_2 x$. Calculate the value of $f(8037776589)$ if $f(x) = x^{g(h(x))}$.

T4. Observe that the set $\{2, 3, 4\}$ has the property that each sum of 2 distinct elements from the set is not divisible by 4 (in other words, $2+3=5$, $2+4=6$, and $3+4=7$ are each not divisible by 4). On the other hand, it is possible to show that any set of 4 consecutive integers has the property that some sum of two of its elements is divisible by 4. Calculate the integer A, such that (i) and (ii) below both hold.

(i) There is a set S consisting of $A-1$ consecutive integers such that each sum of 2 distinct elements of S is NOT divisible by 1994.

(ii) Every set of A consecutive integers is such that $a+b$ is divisible by 1994 for some choice of 2 distinct integers a and b from the set.

T5. A lattice point is a point (a, b) in the plane with a and b both integers. A lattice point is "visible from the origin" if the line segment joining the origin to the point contains no lattice points other than the origin and the point. For example, $(1, 2)$ is visible from the origin but $(2, 2)$ is not (the point $(1, 1)$ lies on the segment joining the origin to $(2, 2)$). Find an integer b such that $0 < b \leq 400$ and

$$(2, b),\ (3, b),\ (4, b),\ (5, b),\ (6, b),\ (7, b),\ \text{ and } (8, b)$$

are all NOT visible from the origin.

T6. Calculate the sum of the squares of the *real* roots of

$$x^{20} - x^{18} - x^{16} - x^{14} - \cdots - x^4 - x^2 - 2.$$

T7. Calculate the largest integer which is less than

$$\sqrt{2500} - \sqrt{2501} + \sqrt{2502} - \sqrt{2503} + \cdots - \sqrt{2999} + \sqrt{3000}.$$

T8. Let $a_2, a_3, \ldots, a_{100}$ be defined as follows:

$$\begin{aligned} a_{100} &= 100 \\ a_{99} &= 99^{a_{100}} = 99^{100} \\ a_{98} &= 98^{a_{99}} = 98^{99^{100}} \\ &\vdots \end{aligned}$$

Thus, in general, $a_n = n^{a_{n+1}}$ for $2 \leq n \leq 99$. Calculate the units digit of a_2.

T9. The graphs of $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect in exactly 3 distinct points. Calculate the slope of the line passing through 2 of these points.

T10. In the drawing below, the equation of the circle is $x^2 + y^2 = 1$. Also, $\angle BDA = 17°$ and the distance from A to B is equal to the reciprocal of the distance from A to C. Calculate the $x-$coordinate of C? (Comment: The measurements in the drawing may not be accurate.)