TEAM PROBLEMS February, 2001

Instructions: Answer as many of the problems below as you can. At the end of the time allotted, turn in a list of your answers. Your answers should be expressed in simplest form. Exact answers are required on all problems.

1. What is the maximum value of the function

$$3\cos\theta - 2\sin\theta\,?$$

2. There are 6 people in a room. Each person randomly chooses a positive integer ≤ 20. What is the probability that some two of the people choose the same number? Express your answer as an exact decimal.

3. The largest *known* explicit example of a prime number is currently $2^{6972593}-1$. How many digits does it have?

4. Let P be a point in an equilateral triangle with each side of length 1. Let h_1, h_2, and h_3 be the distances from P to the three sides of the triangle. What are all the possible values for $h_1 + h_2 + h_3$?

5. Suppose that $f(x)$ is a polynomial of degree 5 and with leading coefficient 2001. Suppose further that

$$f(1) = 1, \quad f(2) = 3, \quad f(3) = 5, \quad f(4) = 7, \quad \text{and} \quad f(5) = 9.$$

What is the value of $f(6)$?

6. Suppose x and y are real numbers such that

$$2x^2 + y^2 - 2xy + 12y + 72 \leq 0.$$

What is the value of x^2y?

Solutions are located at the website http://www.math.sc.edu/~filaseta/contests/contests.html

7. A triangle has two sides of length 5 and one side of length 6. A rectangle R is formed with one edge on the side of length 6 and a vertex (or corner point) on each of the other two sides. What is the maximum possible value for the area of R?

8. Let
$$N = 10^{96} - 10^{80} + 10^{64} - 10^{48} + 10^{32} - 10^{16} + 1.$$
Then
$$\frac{1}{N} = 0.\overline{d_1 d_2 d_3 \ldots d_r}$$
(so that the block of digits $d_1 d_2 d_3 \ldots d_r$ repeats indefinitely). There is more than one value of r for which such digits $d_1, d_2, \ldots, d_r$ exist. If the smallest such r is k, then it is known (and not so hard to show) that every such value of r is divisible by k. What is the value of k?