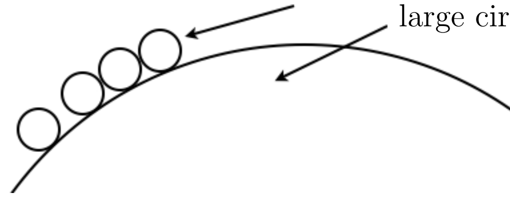
Team Problems March 5, 2005

Instructions: Answer as many of the problems below as you can. At the end of the time allotted, turn in a list of your answers. Your answers should be expressed in simplest form.

1. Circles of radius one unit each are placed around the outside and tangent to a circle of radius 2005 units with no two of the smaller circles overlapping. What is the maximum number of unit circles that can be used?



2. The equation

$$\cos(18\theta) = 131072\cos^{18}\theta - 589824\cos^{16}\theta + 1105920\cos^{14}\theta - 1118208\cos^{12}\theta + 658944\cos^{10}\theta - 228096\cos^{8}\theta + 44352\cos^{6}\theta - 4320\cos^{4}\theta + 162\cos^{2}\theta - 1,$$

is a valid trigonometric identity. Determine the exact value of $\displaystyle\prod_{k=1}^{9}\cos\big((2k-1)\cdot 5^\circ\big)$.

3. What is the greatest common divisor of the 2010 digit and 2005 digit numbers below?

$$\underbrace{2222222\ldots22222}_{2010\ \ 2\text{'s}} \qquad \underbrace{77777\ldots7777}_{2005\ \ 7\text{'s}}$$

4. What is the coefficient of x^{2005} in

$$(x+1)^7(x^2+1)^4(x^4+1)^5(x^8+1)(x^{16}+1)(x^{32}+1)(x^{64}+1)(x^{128}+1)(x^{256}+1)(x^{512}+1)(x^{1024}+1)\,?$$

5. Find the smallest positive angle θ in *degrees* satisfying

$$\sin^2(2004\,\theta) + \cos^2(2005\,\theta) = 1.$$

Express your answer as a reduced rational number of degrees.

Solutions are located at the website `http://www.math.sc.edu/~filaseta/contests/contests.html`

6. Let $A = (0, 0)$ and $B = (100, 0)$. For each point C in the plane with the area of $\triangle ABC$ equal to 2005, consider the point D on line $\overleftrightarrow{BC}$ with segment $\overline{AD}$ an altitude to $\triangle ABC$. The set of all such points D together with the point A enclose a region in the plane. What is the area of this region? Give an exact answer.

7. Consider the powers of 2, beginning with 2, which have a leading digit 1. The first few are $2^4 = 16$, $2^7 = 128$, $2^{10} = 1024$, and $2^{14} = 16384$. How many such powers of 2 are there $\leq 2005^{2005}$?

8. In $\triangle ABC$ (not drawn to scale), the altitude from A, the angle bisector of $\angle BAC$, and the median from A to the midpoint of $\overline{BC}$ divide $\angle BAC$ into four equal angles. What is the measure in degrees of angle $\angle BAC$?