SOLUTIONS TO TEAM PROBLEMS
November 15, 2003

Answers:

1. 10.9679
2. 0.457427
3. 5
4. 2145, 2320, 2385, 2400, 2401, & 2465
5. 0.0155
6. 45.25768...
7. 301
8. 1/2001

1. As $\log_b a = \log a / \log b$ (where $\log x$ can be the logarithm to any fixed base), we deduce that

$$(\log_2 3)(\log_3 4)\cdots(\log_{2002} 2003) = \frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \frac{\log 5}{\log 4} \times \cdots \times \frac{\log 2002}{\log 2001} \times \frac{\log 2003}{\log 2002}$$
$$= \frac{\log 2003}{\log 2} = 10.9679467\ldots.$$

2. Let G be the midpoint of segment $\overline{BC}$ so line $\overleftrightarrow{DG}$ is perpendicular to line $\overleftrightarrow{BC}$. Since $\triangle BDG$ is a $30-60-90$ degrees triangle with $BG = 1/2$, we see that $DG = 1/(2\sqrt{3})$. Since $DE = 1$ and $\triangle DGE$ is a right triangle, we deduce $GE = \sqrt{DE^2 - DG^2} = \sqrt{1-(1/12)} = \sqrt{11/12}$. Since $GC = 1/2$, we obtain $CE = \sqrt{11/12} - 1/2 = 0.457427\ldots.$

3. The first person should begin by moving A to the rectangle numbered 5. In fact, the first person can win by, on each turn, putting A on a rectangle along the diagonal from the rectangle labelled B to the rectangle numbered 5. The second player will have to move off this diagonal and then the first player can continue to move on the diagonal. Since the rectangle labelled with B is on the diagonal, the first person will eventually win (with this strategy). Observe that if first player does not put A on the rectangle numbered 5, then the second player can force a win by using the above strategy (putting A along the diagonal).

4. We may suppose that $x \geq 0$ and $y \geq 0$ and do so. Clearly, we must have $y < x$. Observe that $6^4 < 2003 < 7^4$, so $x \geq 7$. If $x \geq 10$, then $x^4 - y^4 \geq 10^4 - 9^4 > 3000$. So $x \in \{7, 8, 9\}$. Since $7^4 - 5^4 < 2000$, if $x = 7$, we must have $0 \leq y \leq 4$. Trying these values of y, we see that $x^4 - y^4$ can equal 2145, 2320, 2385, 2400, and 2401. Since $8^4 - 7^4 = 1695 < 2003$ and $8^4 - 6^4 = 2800 > 2500$, there are no N satisfying the conditions in the problem when $x = 8$. Since $9^4 - 8^4 = 2465$ and $9^4 - 7^4 > 4000$, 2465 is the only N satisfying the conditions in the problem when $x = 9$. The answer is 2145, 2320, 2385, 2400, 2401, and 2465.

5. There are $\lfloor 100/11 \rfloor = 9$ positive multiples of 11 that are ≤ 100, $\lfloor 100/13 \rfloor = 7$ positive multiples of 13 that are ≤ 100, $\lfloor 100/23 \rfloor = 4$ positive multiples of 23 that are ≤ 100, and $\lfloor 100/31 \rfloor = 3$ positive multiples of 31 that are ≤ 100. We deduce that $\gcd(a,b)$ is divisible by 11 precisely when each of a and b is divisible by 11 which occurs for $9^2 = 81$ pairs (a,b). Similarly, $\gcd(a,b)$ is divisible by 13 for exactly $7^2 = 49$ pairs (a,b), $\gcd(a,b)$ is divisible by 23 for exactly $4^2 = 16$ pairs (a,b), and $\gcd(a,b)$ is divisible by 31 for exactly $3^2 = 9$ pairs (a,b). Observe that there are no positive integers ≤ 100 divisible by two of 11, 13, 23, and 31. Hence, there are exactly $81+49+16+9 = 155$ pairs (a,b) with $\gcd(a,b)$ divisible by at least one of 11, 13, 23, and 31. Thus, the probability is $155/100^2 = 0.0155$.

6. Let $f(x)$ be the polynomial in the problem. Thus, $f(x) = (x^2 - x - 2003)x^{2001} - x^2 - 2003$. Let
$$\alpha = (1+\sqrt{8013})/2 = 45.25768\ldots$$
so that α is a root of the quadratic $x^2 - x - 2003$. Clearly, $f(\alpha) < 0$. One checks directly that $f(45.257681)$ is positive (and, in fact, $> 10^{3000}$). It follows that $f(x)$ must have a root between 45.25768 and 45.257681. Thus, the answer is $45.25768\ldots$.

7. Observe that
$$\begin{aligned}\sqrt{2^{2004}+1} - 2^{1002} &= \frac{1}{\sqrt{2^{2004}+1}+2^{1002}} \in \left(\frac{1}{2.1\cdot 2^{1002}}, \frac{1}{2\cdot 2^{1002}}\right) \\ &\subseteq \left(1.1110281\ldots\cdot 10^{-302}, 1.1665795\ldots\cdot 10^{-302}\right).\end{aligned}$$
Thus, the answer is 301.

8. For $2 \le n \le 2005$, we use that
$$\begin{aligned}\frac{(n-1)!}{(n+2001)!} &= \frac{1}{n(n+1)(n+2)\cdots(n+2001)} \\ &= \frac{1}{2001}\left(\frac{1}{n(n+1)\cdots(n+2000)} - \frac{1}{(n+1)(n+2)\cdots(n+2001)}\right).\end{aligned}$$
The sum in the problem is therefore
$$\frac{1}{2001}\sum_{n=2}^{2005}\left(\frac{1}{n(n+1)\cdots(n+2000)} - \frac{1}{(n+1)(n+2)\cdots(n+2001)}\right).$$
This is a telescoping sum with the second part of each term cancelling with the first part of the next (except in the case of the last term). Hence, the sum is
$$\frac{1}{2001}\left(\frac{1}{2002!} - \frac{2005!}{4006!}\right).$$
The answer is therefore $1/2001$ (and this answer is unique).