SOLUTIONS TO TEAM PROBLEMS
March 5, 2005

Answers:

1. 6302
2. $1/\sqrt{131072}$ or $1/(256\sqrt{2})$ or $\sqrt{2}/512$
3. 77777
4. 8192
5. $(180/4009)^\circ$ or $180/4009$
6. 2500π
7. 6620
8. 90°

1. Consider a tight fitting of the smaller unit circles around the larger circle of radius 2005 so that each smaller circle is tangent to two adjacent circles and the larger circle. Let A and B denote the centers of some two adjacent circles, and let C denote the center of the circle with radius 2005. Then $\triangle ABC$ is isosceles. Letting D be the point on $\overline{AB}$ with $\overline{AD}$ an altitude of $\triangle ABC$, we see that $\triangle ADC$ is a right triangle, and $\sin\angle ACD = 1/2006$. We deduce that $\angle ACD = \sin^{-1}(1/2006) = 0.0285622^\circ$. It follows that $\angle ACB = 0.0571244\ldots^\circ$. With this tight fit, the number of smaller circles around the larger circle is, therefore, the greatest integer $\le 360/0.0571244\cdots = 6302.035557\ldots$. This implies that we can fit 6302 unit circles and no more around the larger circle of radius 2005 with some room to spare (so that, in particular, if you do not want the *nonoverlapping* unit circles to even touch at a point, we have room to move them ever so slightly so that this is the case). The answer is 6302. (Note that the computations above are approximations that should be dealt with carefully. The number $6302.035557\ldots$ should really be replaced by $\pi/\sin^{-1}(1/2006) = 6302.0346\ldots$.)

2. Letting $\theta = (2k-1)\cdot 5^\circ$ for k an integer in $[1,9]$, we see that $0 < \theta < 90^\circ$ and $\cos(18\theta) = 0$. From the information given, we deduce that the 9 numbers $\cos\big((2k-1)\cdot 5^\circ\big)$ with $1 \le k \le 9$ are distinct positive real roots of

$$f(x) = 131072x^{18} - 589824x^{16} + 1105920x^{14} - 1118208x^{12} + \cdots - 4320x^4 + 162x^2 - 1.$$

The polynomial $f(x)$ above is merely the expression on the right-hand side of the display in the problem with $\cos\theta$ replaced with x so that, in particular, $f(x)$ is a polynomial in x^2 (i.e., each exponent of x is even). We deduce that $-\cos\big((2k-1)\cdot 5^\circ\big)$ with $1 \le k \le 9$ are distinct negative real roots of $f(x)$. As $f(x)$ is of degree 18, we obtain that the 18 numbers $\cos\big((2k-1)\cdot 5^\circ\big)$ and $-\cos\big((2k-1)\cdot 5^\circ\big)$ with $1 \le k \le 9$ are all the roots of $f(x)$. If P denotes the product in the problem, we see that $-P^2$ is the product of these 18 roots. On the other hand, the product of these roots must also be the constant coefficient of $f(x)$ divided by the leading coefficient of $f(x)$. Hence, $-P^2 = -1/131072$ so that $P = 1/\sqrt{131072} = 1/(256\sqrt{2}) = \sqrt{2}/512$.

3. Let A denote the 2010 digit number that is a string of 2's, and let B denote the 2005 digit number that is a string of 7's. We consider also the numbers $A' = A/2$ and $B' = B/7$,

each of which is a string of 1's. Observe that B is odd, so 2 does not divide B. On the other hand, the divisor 7 of B is a divisor of A as can be seen as follows. The number 111111 is divisible by 7 so that any string of a multiple of 6 number of 1's is also divisible by 7 (for example, $111111111111111111 = 111111 \cdot 10^{12} + 111111 \cdot 10^{6} + 111111$ is divisible by 7 since each term on the right is). As 2010 is a multiple of 6, we deduce A' and, hence, A is divisible by 7. Thus, 7 is a common factor of A and B. Now, the equation $A' - 10^5 B' = 11111$ implies any divisor of A' and B' must also be a divisor of 11111. Arguing in a manner similar to before, we see that since 2010 and 2005 are both multiples of 5, the number 11111 divides both A' and B'. Thus, the greatest common divisor of A' and B' is 11111. Note that 7 does not divide 11111. We obtain from the definition of A' and B' that the greatest common divisor of A and B is $7 \cdot 11111 = 77777$.

4. Let $g(x) = (x+1)^6(x^2+1)^3(x^4+1)^4$, and let

$$\begin{aligned} h(x) &= (x+1)(x^2+1)(x^4+1)\cdots(x^{256}+1)(x^{512}+1)(x^{1024}+1) \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^{2046} + x^{2047}, \end{aligned}$$

where the last equation follows by using that each positive integer can be written uniquely as a sum of powers of 2. If $f(x)$ is the polynomial in the problem, then $f(x) = g(x)h(x)$, so we want to know the coefficient of x^{2005} in the product $g(x)h(x)$. Writing $g(x) = b_{28}x^{28} + b_{27}x^{27} + \cdots + b_2x^2 + b_1x + b_0$, we see that the coefficient of x^{2005} in $g(x)h(x)$ can be obtained by taking each b_k (the coefficient of x^k in $g(x)$) times 1 (the coefficient of x^{2005-k} in $h(x)$) and adding these numbers together. In other words, the answer to the problem is $b_{28} + b_{27} + \cdots + b_2 + b_1 + b_0$. On the other hand, this number is just $g(1) = 2^6 \cdot 2^3 \cdot 2^4 = 2^{13} = 8192$.

5. Using that $1 - \sin^2(2004\,\theta) = \cos^2(2004\,\theta)$, the given equation can be rewritten as $\cos^2(2005\,\theta) = \cos^2(2004\,\theta)$ which is equivalent to $\cos(2005\,\theta) = \pm\cos(2004\,\theta)$. As $\cos x$ is decreasing for $x \in [0, \pi]$, we have $\cos(2005\,\theta) \neq +\cos(2004\,\theta)$ for $\theta \in (0, \pi/2005]$. We find the $\theta \in (0, \pi/2005]$ for which $\cos(2005\,\theta) = -\cos(2004\,\theta)$. Equality occurs if and only if $2005\,\theta = \pi/2 + t$ and $2004\,\theta = \pi/2 - t$ for some positive t (in the appropriate range). Solving for θ (for example, by adding these equations) we obtain $\theta = \pi/4009$. Hence, the answer (converting to degrees as requested) is $(180/4009)^\circ$.

6. For each D constructed in the problem, $\angle ADB = 90^\circ$ so that the point D lies on the circle with diameter $\overline{AB}$. As the line through A and B rotates about the point B, there are two points C on the line for which $\triangle ABC$ has area 2005 and a unique point D on line $\overleftrightarrow{BC}$ (independent of the C chosen) such that $\overline{AD}$ is an altitude to $\triangle ABC$. As the line rotates, the curve traced out by D is a complete circle except the point A is missing (when the line being rotated is on the x-axis). The set of points D together with the point A, therefore, form a circle with radius 50. The area of the circle, and the answer to the problem, is $\pi \cdot 50^2 = 2500\pi$. Note that 2005 may be replaced by any positive real number without affecting the answer to the question; the described region is still the same circle independent of the fixed value for the area.

7. The positive integers having leading digit 1 are precisely the integers in the intervals $[10^k, 2 \cdot 10^k)$ where $k \in \{0, 1, 2, \dots\}$. For each such k, there is exactly one power of 2 lying in the interval $[10^k, 2 \cdot 10^k)$ (to see this for $k \geq 1$, consider the largest power of 2 that is $< 10^k$ and double it; convince yourself that this number is a power of 2 in $[10^k, 2 \cdot 10^k)$ and that the next power of 2 is outside this interval). As the problem concerns powers of 2 beginning with 2, we consider only $k \geq 1$. We are interested in the number of intervals $[10^k, 2 \cdot 10^k)$ with $2 \cdot 10^k \leq 2005^{2005}$ but also need to consider whether 2005^{2005} itself lies in such an interval. As

$$\log_{10}\left(2005^{2005}\right) = 2005 \log_{10} 2005 = 6620.7393\dots,$$

we see that $2005^{2005} = 10^{6620} 10^{0.7393\dots} = 5.486\dots \cdot 10^{6620}$. We deduce that 2005^{2005} does not have leading digit 1 (so it is not in an interval of the from $[10^k, 2 \cdot 10^k)$) and that the k of interest to us are the positive integers ≤ 6620. Hence, the answer is 6620.

8. Label the points along $\overline{BC}$ that are on the altitude from A, on the angle bisector of $\angle BAC$, and on the median from A as D, E and F, respectively. Let $\theta = \angle BAD$ so that we are interested in the value of 4θ. The triangle $\triangle BAD$ is a right triangle so that $\angle ABC = 90^\circ - \theta$. Also, $\angle ACB = 90^\circ - 3\theta$. We apply the Law of Sines to the triangles $\triangle ABF$ and $\triangle ACF$. Letting $s = AF$ and $t = BF = FC$, we deduce

$$\frac{\cos\theta}{s} = \frac{\sin(90^\circ - \theta)}{s} = \frac{\sin\angle ABF}{s} = \frac{\sin\angle BAF}{t} = \frac{\sin(3\theta)}{t}$$

and

$$\frac{\cos(3\theta)}{s} = \frac{\sin(90^\circ - 3\theta)}{s} = \frac{\sin\angle ACF}{s} = \frac{\sin\angle CAF}{t} = \frac{\sin\theta}{t}.$$

We obtain that $\cos\theta / \sin(3\theta) = s/t = \cos(3\theta)/\sin\theta$ so that

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\sin(3\theta)\cos(3\theta) = \sin(6\theta).$$

As $4\theta = \angle BAC < 180^\circ$, we have $\theta < 45^\circ$. It follows that $\sin(2\theta) = \sin(6\theta)$ only if $2\theta = 180^\circ - 6\theta$. Therefore, $4\theta = 90^\circ$.