SOLUTIONS TO TEAM PROBLEMS
JANUARY, 2003

Answers: 1. A < B < C < D  3. 233
2. Alan shakes 2 hands  4. 6
Alice shakes 2 hands  5. GOOD or GO.OD or good or go.od
Bernice shakes 4 hands  6. 1/9
Bob shakes 0 hands  7. 3
Calvin shakes 3 hands  8. log 2 or log_e 2 or ln 2
Cathy shakes 1 hand

1. Let \( f(x) = \log \log x \). The base of the logarithm is not important. We will use the natural logarithm here. Since \( f(x) \) is an increasing function, we may consider \( f(A) \), \( f(B) \), \( f(C) \), and \( f(D) \) instead of \( A \), \( B \), \( C \), and \( D \). The number \( A \) is a tower (say) of twelve \( \sqrt{2} \)'s. It is increased by replacing the highest exponent \( \sqrt{2} \) (at the top) with 2. Since \( \sqrt{2}^2 = 2 \), we deduce \( A \leq 2 \) so that \( f(A) \leq \log \log 2 < 0 \). Note that

\[
f(B) = \log \log B = \log \left( 10^{100} \log 10 \right) = 100 \log 10 + \log \log 10 = 231.09\ldots.
\]

Similarly,

\[
f(C) = 2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}} \log 2 + \log \log 2 = 2^{16} \log 2 + \log \log 2 = 45425.727\ldots.
\]

Also, \( f(D) = \sqrt{3}^t \log \sqrt{3} + \log \log \sqrt{3} \) where \( t \) consists of a tower of five \( \sqrt{3} \)'s. The value of \( t \) can be estimated with a calculator by considering

\[
\log t = \sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}}}}} \log \sqrt{3} = 5.359\ldots
\]

Thus,

\[
f(D) > \sqrt{3}^{e^{5.359}} \log \sqrt{3} + \log \log \sqrt{3} > 2 \times 10^{50}.
\]

Thus, \( f(A) < f(B) < f(C) < f(D) \). The answer is \( A < B < C < D \) (no wonder the problem said to put the answer in this form).

2. Let \( h(x) \) denote the number of hands person \( x \) shakes. The most hands that anyone shakes in this problem is 4. Since Alice, Bernice, Bob, Calvin and Cathy shook a different number of hands, they shook 0, 1, 2, 3, and 4 hands in some order. Also, the given information in the problem implies

\[
h(\text{Bob}) \leq 3, \quad h(\text{Alan}) \geq 1, \quad \text{and} \quad 1 \leq h(\text{Calvin}) \leq 3.
\]
Observe that Alice cannot shake 4 hands since then nobody shakes 0 hands. Also, if Cathy shakes 4 hands, then nobody shakes 0 hands. It follows that Bernice shakes 4 hands. From this, we deduce that Bob shakes 0 hands (since Alice, Calvin and Cathy shake hands with Bernice). Since Calvin shakes hands with both Alan and Bernice, $h(\text{Calvin}) \geq 2$. Now, $h(\text{Alice}) \neq 1$ since otherwise each of Calvin and Cathy do not shake hands with either Alice or Bob so that no one shakes 3 hands. Thus, Cathy shakes 1 hand, apparently Bernice's. This means that Alice cannot shake 3 hands so that $h(\text{Alice}) = 2$ and $h(\text{Calvin}) = 3$. One now checks that $h(\text{Alan}) = 2$ (he shakes hands with Bernice and Calvin and nobody else).

3. If an integer $m > 1$ divides $2^{29} - 1$, then so does each of $m$'s prime factors. It follows that the smallest integer $m > 1$ that divides $2^{29} - 1$ is a prime. Taking this prime to be $q$ and taking $a = 2$ and $p = 29$ in the information given in the problem, we deduce that $q - 1$ is not divisible by 59. The numbers 117 and 175 are not prime (the first is divisible by 3 and the second by 5). Finally, one checks directly that $2^{29} - 1$ is divisible by 233, so this is the answer.

4. Let


Since $f(x)$ has degree 2003, the graph of $y = f(x)$ crosses the $x$-axis at most 2003 times. Observe that $f(k) > 0$ for each $k \in \{1, 2, \ldots, 2003\}$. Also,

$$f(0.5) < 0, \ f(2.5) < 0, \ f(4.5) < 0, \ldots, \ f(2002.5) < 0.$$ 

Therefore, the graph of $y = f(x)$ must cross the $x$-axis at least once in each of the intervals $(0.5, 1), (2, 2.5), (2.5, 3), (4, 4.5), (4.5, 5), \ldots, (2002, 2002.5), (2002.5, 2003)$.

Since this consists of 2003 intervals, we deduce that the graph of $y = f(x)$ crosses the $x$-axis exactly once in each of the intervals above and nowhere else. It follows that the graph crosses the $x$-axis 6 times in the interval $[1000, 1005]$.

5. We want to write the number

$$\sqrt{29085} = 170.5432496 \ldots$$

in base 26. Since $170 = 6 \cdot 26 + 14$, the first two digits come from converting 6 and 14 to letters, so they are G and O, respectively. Since $14/26 = 0.53846 \ldots$ and $15/26 = 0.57692 \ldots$, we obtain that the next digit corresponds to 14 and, hence, is the letter O. Next, we use that

$$170.5432496 \cdots - \left( 6 \cdot 26 + 14 + \frac{14}{26} \right) = 0.004788 \ldots.$$ 

Since $3/26^2 = 0.0044 \ldots$ and $4/26^2 = 0.0059 \ldots$, the fourth digit corresponds to 3 and, hence, is the letter D. Thus, the answer is GOOD.
6. Consider the two circles tangent to each other and a line as shown to the right. Suppose the smaller circle has radius $r$ and the larger circle radius $R$. Then the shaded triangle is a right triangle with hypotenuse $R + r$ and one leg of length $R - r$. It follows that if the “horizontal distance between these circles” (that is the length of the remaining leg) is $h$, then

\[ h^2 = (R + r)^2 - (R - r)^2 = 4Rr \implies h = 2\sqrt{Rr}. \]

In the problem, let $h(i, j)$ denote the horizontal distance between $C_i$ and $C_j$. Let $r$ denote the radius of $C_2$. Since $h(0, 1) = h(0, 2) + h(1, 2)$, we deduce from the formula for $h$ above that $2 = 2\sqrt{r} + 2\sqrt{r}$ so that $r = 1/4$. Let $r'$ denote the radius of $C_3$. Since $h(0, 2) = h(0, 3) + h(2, 3)$, we deduce from the formula for $h$ above that $1 = 2\sqrt{r'} + \sqrt{r'}$ so that $r' = 1/9$.

7. Let $f(x) = x^{2003} + ax^{10} + 200$ where $a \in \{1, 2, \ldots, 200\}$. If $z$ is a complex number with $|z| \geq 1.003$, then

\[ |f(z)| \geq |z|^{2003} - a|z|^{10} - 200 = |z|^{10}(|z|^{1993} - a) - 200 \geq 1.03(391.525 - a) - 200. \]

For $a \leq 197$, this last expression is $> 0$ so that $f(z) \neq 0$. In other words, $f(x)$ has no roots with absolute value $\geq 1.003$ unless $a > 197$. For $a \in \{198, 199, 200\}$, one checks directly that the value of $f(-1.003) > 0$ and $f(-2) < 0$ so that $f(x)$ in fact has a negative real root between $-1.003$ and $-2$. The answer, therefore, is $3$.

8. One can use that if $n$ is large, then $S_n \approx \log n$ and $S_{2n} \approx \log(2n) = \log n + \log 2$ so that $T_n = S_{2n} - S_n \approx \log 2$. This idea is not a true explanation, though, as we are not given sufficient information in the problem to determine how good an approximation $\log n$ is to $S_n$. We give a more precise explanation as follows. Let $c$ denote the constant that $T_n$ approaches as $n$ tends to infinity. Let $k$ be a positive integer. We use that

\[ 1 + T_1 + T_2 + T_4 + T_8 + \cdots + T_{2^{k-1}} = S_{2^k}. \]

We rewrite this as

\[ \frac{1 + T_1 + T_2 + T_4 + T_8 + \cdots + T_{2^{k-1}}}{k + 1} \cdot \frac{k + 1}{k} \cdot \frac{1}{\log 2} = \frac{S_{2^k}}{k \log 2} = \frac{S_{2^k}}{\log (2^k)}. \]

Observe that the first fraction on the left is simply the average of $1, T_1, T_2, T_4, \ldots, T_{2^{k-1}}$ and, as $k$ gets large, this average approaches $c$. Also, as $k$ gets large, $(k + 1)/k$ and the last fraction above approach $1$ (the latter by what is given). We deduce that $c/\log 2 = 1$ so that $c = \log 2$. 

\[ 1 + T_1 + T_2 + T_4 + T_8 + \cdots + T_{2^{k-1}} = S_{2^k}. \]