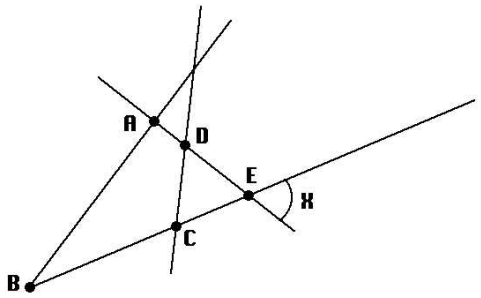
TEAM PROBLEMS FEBRUARY, 2002

Instructions: Answer as many of the problems below as you can. At the end of the time allotted, turn in a list of your answers. Your answers should be expressed in simplest form. Exact answers are required on all problems unless specified otherwise.

1. In the drawing (not to scale) at the right, $DC = DE$, $\angle ABC = 28°$, and the points A, B, C, and D all lie on the same circle (not drawn). What is the value of x (in degrees)?



2. Find the largest and smallest values of $\sin\left((2^n)^\circ\right)$ where n is an integer satisfying $0 \le n \le 2002$. Express your answers accurate to five decimal digits.

3. Describe with one word the graph of $x^2 + xy + y^2 = 0$ in the xy-plane. Be precise. For example, "square" is precise enough but wrong, "graph" is not precise enough (and is otherwise correct), and "round" is not precise enough (and may or may not be otherwise correct).

4. The number $10^{2002} - 1$ is divisible by 2003. What are the 11111111^{th} and 11111112^{th} digits after the decimal in the decimal expansion of $1/2003$?

5. The product

$$\left(x - 2^2\right)\left(x - 2^{2^2}\right) \cdots \left(x - 2^{2^{1001}}\right)\left(x + 2^{2^{1002}}\right)\left(x + 2^{2^{1003}}\right) \cdots \left(x + 2^{2^{2002}}\right)$$

is a polynomial of degree 2002. The first 1001 factors above contain a negative term, and the last 1001 factors do not. Interpret 2^{2^n} as $2^{(2^n)}$. When the product is expanded, it has 2003 coefficients (counting the constant term). How many of these coefficients are positive?

Solutions are located at the website `http://www.math.sc.edu/~filaseta/contests/contests.html`

6. The polynomial $x^4 + x^3 + x^2 + x + 1$ has one root of the form $a + b\mathrm{i}$ where a and b are positive real numbers and $\mathrm{i}^2 + 1 = 0$. If

$$ar + bs = \cos(3^\circ) \qquad \text{and} \qquad r = \cos(75^\circ),$$

what is the exact value of $r + s$? Express your answer in a form that does not use trigonometric functions.

7. It is known that the largest integer n such that each prime factor of $n(n+1)$ is ≤ 11 is $n = 9800$. What is the largest integer y having each prime factor ≤ 11 and satisfying the equation $x^2 - 2y^2 = 1$ for some integer x?

8. A person is said to be n years old, where n is a nonnegative integer, if the person has lived at least n years and has not yet lived $n + 1$ years. At some point in time, Tammy is 4 years old and John is three times as old as Martha (so if John is j years old and Martha is m years old, then j and m are nonnegative integers with $j = 3m$). At some other time, Martha is twice as old as Tammy and John is five times as old as Tammy. At yet another time, John is twice as old as Martha and Tammy is T years old. There are different possibilities for what T can be. What is the largest possible value of T?