On the nonlinearity of quantum dynamical entropy

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The entropy function

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Fact 1

The entropy function, η, has the following properties.

- *(Nonnegative)* $\eta(x) \geqslant x$ *for all* $x \in [0,1]$.
- *(Strictly concave)* $\eta(pt + qs) > t\eta(p) + s\eta(q)$, *for all* $t, s \in [0,1]$ *and* $p \in (0,1)$, *where* $q = 1 - p$.
- *(Countable subadditivity)* $\eta(\sum_{n=1}^{\infty} t_n) \leqslant \sum_{n=1}^{\infty} \eta(t_n)$, *whenever* $\{t_n\}_{n=1}^{\infty} \subseteq [0,1]$.

Partitions

Let (Ω, Σ, μ) be a probability space. We say (Ω, Σ, μ) is *discrete* if Ω is countable and $\Sigma = \mathcal{P}(\Omega)$. We denote by $\mathcal{P}_{ar}(\Omega)$ the lattice of countable and measurable partitions of Ω.

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Let $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{ar}(\Omega)$.

- We say that $\mathcal{C}$ is finer than $\mathcal{D}$, and write $\mathcal{D} \leqslant \mathcal{C}$, if, for every $D \in \mathcal{D}$, there exists $\mathcal{C}_D \subseteq \mathcal{C}$ such that $D = \cup \mathcal{C}_D$.

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- The join of $\mathcal{C}$ and $\mathcal{D}$ is given by $\mathcal{C} \vee \mathcal{D} := \{C \cap D : C \in \mathcal{C} \text{ and } D \in \mathcal{D}\} \in \mathcal{P}_{ar}(\Omega)$.

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- Whenever Ω is discrete, we call the partition into singletons, $\{\{\omega\}\}_{\omega \in \Omega} \in \mathcal{P}_{ar}(\Omega)$, the atomic partition and denote it by $\mathcal{A}$.

Entropy of partitions

For $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, the *entropy* of $\mathcal{C}$ is given by

$$H(\mathcal{C}) = \sum_{A \in \mathcal{C}} \eta(\mu(A)).$$

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Fact 2

For all $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{ar}(\Omega)$, we have the following:

- $0 \leqslant H(\mathcal{C})$ *with equality iff there exists an* $A \in \mathcal{C}$ *such that* $\mu(A) = 1$.
- $H(\mathcal{C}) \leqslant \log |\mathcal{C}|$ *with equality (in the case* $|\mathcal{C}| < \infty$*) iff* $\mu(A) = \frac{1}{|\mathcal{C}|}$ *for all* $A \in \mathcal{C}$.
- $H(\mathcal{D}) \leqslant H(\mathcal{C})$ *whenever* $\mathcal{D} \leqslant \mathcal{C}$.

Conditional entropy of partitions

The *conditional entropy* of $\mathcal{C}$ given $\mathcal{D}$ is given by

$$H(\mathcal{C}|\mathcal{D}) := \sum_{D\in\mathcal{D}} \mu(D) \sum_{C\in\mathcal{C}} \eta(\mu(C|D)).$$

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Fact 3

For all $\mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathcal{P}_{ar}(\Omega)$, we have the following:

- *(Chain Rule)*

  $$H(\mathcal{C} \vee \mathcal{D}) = H(\mathcal{D}) + H(\mathcal{C}|\mathcal{D})$$

  or more generally

  $$H(\vee_{k=0}^{n} \mathcal{C}_n) = H(\mathcal{C}_0) + \sum_{k=1}^{n} H(\mathcal{C}_k | \vee_{\ell=0}^{k-1} \mathcal{C}_\ell)$$

- $H(\mathcal{C}|\mathcal{D}) \geqslant 0$ *with equality iff* $\mathcal{C} \leqslant \mathcal{D}$.
- $H(\mathcal{C}|\mathcal{D}) \leqslant H(\mathcal{C}|\mathcal{B})$ *whenever* $\mathcal{B} \leqslant \mathcal{D}$.

Dynamical entropy of partitions

Theorem 4

Let $(\mathcal{C}_n)_{n=0}^{\infty} \subseteq \mathcal{P}_{ar}(\Omega)$ *be a sequence of partitions. If* $\lim_{n\to\infty} H(\mathcal{C}_n | \vee_{k=0}^{n-1} \mathcal{C}_k)$ *exists, then* $\lim_{n\to\infty} \frac{1}{n} H(\vee_{k=0}^{n-1} \mathcal{C}_k)$ *exists and the limits are equal.*

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Proof.

The chain rule gives that $H(\vee_{k=0}^{n-1}\mathcal{C}_k) = H(\mathcal{C}_0) + \sum_{k=1}^{n-1} H(\mathcal{C}_n | \vee_{\ell=0}^{k-1} \mathcal{C}_\ell)$, for all $n \in \mathbb{N}$. The proof then follows from the Césaro mean Theorem. □

Dynamical systems

Let (Ω, Σ, μ) be a probability space and $f : \Omega \to \Omega$ be a measurable map. The quadruple (Ω, Σ, μ, f) is called a *dynamical system*. If, for all $A \in \Sigma$, $\mu(A) = \mu(f^{-1}(A))$ we say that μ is f-invariant and call the dynamical system (Ω, Σ, μ, f) *stationary*.

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Remark

In the literature, the quadruple (Ω, Σ, μ, f) is referred to as a dynamical system only in the case that μ is f-invariant.

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Fact 5

Let $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$ and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then, for all $n \in \mathbb{N}_0$, we have

- $f^{-n}(\mathcal{C}) := \{f^{-n}(A)\}_{A \in \mathcal{C}} \in \mathcal{P}_{ar}(\Omega)$ *and*
- $\vee_{k=0}^{n} f^{-k}(\mathcal{C}) = \{f^{-n}(A_n) \cap \cdots \cap f^{-1}(A_1) \cap A_0 | A_0, \ldots, A_n \in \mathcal{C}\}$.

Kolmogorov-Sinai entropy

The *Kolmogorov-Sinai (KS) entropy* of (Ω, Σ, μ, f) with respect to $\mathcal{C}$ is given by

$$h^{KS}(f, \mathcal{C}) = \lim_{n \to \infty} \frac{1}{n} H(\vee_{k=0}^{n-1} f^{-k}(\mathcal{C})),$$

whenever the limit exists.

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Corollary 6

Let (Ω, Σ, μ, f) be a stationary dynamical system and $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$. Then both limits above exist and are equal.

Kolmogorov-Sinai entropy

The KS entropy of (Ω, Σ, μ, f) is given by

$$h^{KS}(f) = \sup_{\substack{\mathcal{C} \in \mathcal{P}_{ar}(\Omega) \\ H(\mathcal{C}) < \infty}} h^{KS}(f, \mathcal{C}).$$

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Theorem 7 (Kolmogorov-Sinai Theorem)

Let (Ω, Σ, μ, f) be a dynamical system and $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{ar}(\Omega)$. If $\sigma(\mathcal{D}) \subseteq \sigma(\cup_{n=0}^{\infty} \vee_{k=0}^{n} f^{-k}(\mathcal{C}))$, then

$$h^{KS}(f, \mathcal{C}) \geqslant h^{KS}(f, \mathcal{D}).$$

In particular, if $\mathcal{C}$ is a generating partition; i.e. $\sigma(\cup_{n=0}^{\infty} \vee_{k=0}^{n} f^{-k}(\mathcal{C})) = \Sigma$, and $H(\mathcal{C}) < \infty$, then $h^{KS}(f) = h^{KS}(f, \mathcal{C})$.

Random variables and stochastic processes

Let (Ω, Σ, μ) be a probability space and $(E, \mathcal{E})$ be a measurable space. An (Ω, E) random variable is a measurable map $X : \Omega \to E$. A sequence, $\mathbf{X} := (X_n)_{n=0}^{\infty}$, of (Ω, E) random variables is an (Ω, E) stochastic process. We call X (or $\mathbf{X}$) *discrete* if its range, E, is discrete. Let p_X and $p_{\mathbf{X}}$ denote, respectively, the probability mass functions (pmfs) for discrete random variable, X, and stochastic process, $\mathbf{X}$; i.e.

$$p_X(x) = \mu(X = x) \text{ and } p_{\mathbf{X}}(x_0, \ldots, x_n) = \mu(\cap_{k=0}^{n}(X_k = x_k))$$

for all $n \in \mathbb{N}_0$. We will denote both by p when there is no confusion.

Stationary and Markov processes

Recall that a discrete (Ω, E) stochastic process, $\mathbf{X}$, is called *stationary* whenever

$$\mu(X_0 = x_0, \ldots, X_n = x_n) = \mu(X_l = x_0, \ldots, X_{n+l} = x_n),$$

for all $n, l \in \mathbb{N}_0$ and $x_0, \ldots, x_n \in E$.

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for all $n, l \in \mathbb{N}_0$ and $x_0, \ldots, x_n \in E$.
We will also consider a discrete Markov process governed by a stochastic matrix P; i.e.

P has (x, y)-entry $p_{x,y} = \mu(X_{n+1} = x | X_n = y)$, for all $n \in \mathbb{N}_0$ and $x, y \in E$.

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Furthermore, we will write p_{X_0} as a probability vector and define Pp_{X_0} by matrix multiplication. We say that p_{X_0} (or $\mathbf{X}$) is P-invariant whenever $Pp_{X_0} = p_{X_0}$.

Entropy in information theory

The partition generated by X, $\mathcal{C}_X \in \mathcal{P}_{ar}(\Omega)$, is given by $\mathcal{C}_X := \{X^{-1}(\{e\}) : e \in E\} \in \mathcal{P}_{ar}(\Omega)$. The *Shannon entropy* of X is given by

$$H(X) := H(\mathcal{C}_X) = \sum_{x \in E} \eta(p(x)) \text{ or more generally}$$

$$H(X_0, \ldots, X_n) := H(\vee_{k=0}^{n} \mathcal{C}_{X_k}) = \sum_{\substack{x_k \in E \\ 0 \leqslant k \leqslant n}} \eta(p(x_0, \ldots, x_n)).$$

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For all $n \in \mathbb{N}$, the *conditional entropy* of X_n given $X_0, \ldots, X_{n-1}$

$$H(X_n | X_0, \ldots, X_{n-1}) := H(\mathcal{C}_{X_n} | \vee_{k=0}^{n-1} \mathcal{C}_{X_k})$$

$$= \sum_{\substack{x_k \in E \\ 0 \leqslant k \leqslant n-1}} p(x_0, \ldots, x_{n-1}) \sum_{x_n \in E} \eta(p(x_n | x_0, \ldots, x_{n-1})).$$

Entropy rate

Let $\mathbf{X} = (X_n)_{n=0}^{\infty}$ be a discrete stochastic process. The *entropy rate* of $\mathbf{X}$ is given by

$$H(\mathbf{X}) := \lim_{n\to\infty} \frac{1}{n} H(X_0, \ldots, X_{n-1}),$$

whenever the limit exists.

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whenever the limit exists. By Theorem 4, we have

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Corollary 8

If $\mathbf{X}$ is a stationary stochastic process, then both limits above exist and are equal.

Entropy rate for discrete Markov processes

Theorem 9

Let $\mathbf{X}$ *be a discrete* (Ω, E) *Markov process governed by the transition matrix* P*. Then* $H(\mathbf{X}) = \lim_{n\to\infty} \sum_{y\in E} (P^n \mu)_y \sum_{x\in E} \eta(p_{x,y})$, *whenever the limit exists. Moreover, if* $\mathbf{X}$ *is stationary then*

$$H(\mathbf{X}) = \sum_{y\in E} \mu_y \sum_{x\in E} \eta(p_{x,y}).$$

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$$H(\mathbf{X}) = \sum_{y\in E} \mu_y \sum_{x\in E} \eta(p_{x,y}).$$

Whenever P has a unique invariant measure, μ, we define the *entropy* of P to be

$$H(P) = \sum_{y\in E} \mu_y \sum_{x\in E} \eta(p_{x,y}).$$

Path space for stochastic processes

Let $\mathbf{X}$ be an (Ω, E) stochastic process. Consider the measurable space $(E^*, \mathcal{E}^*)$, where $E^* := E^{\mathbb{N}_0}$ and $\mathcal{E}^* := \sigma(\cup_{n=0}^{\infty} \mathcal{E}^n)$. For all $n \in \mathbb{N}_0$, collection of integer times $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $A_0, \ldots, A_n \in \mathcal{E}$, we define the cylinder set

$$C \begin{pmatrix} A_0 & \cdots & A_n \\ t_0 & \cdots & t_n \end{pmatrix} := \{x = (x_i)_{i \in \mathbb{N}_0} \in E^* : x_{t_k} \in A_k \text{ for } k \in \{0, \ldots, n\}\}.$$

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The measure generated by $\mathbf{X}$, $\mu^{\mathbf{X}}$, is given on each cylinder set by

$$\mu^{\mathbf{X}}(C \begin{pmatrix} A_0 & \cdots & A_n \\ t_0 & \cdots & t_n \end{pmatrix}) = \mu(\cap_{k=0}^{n} (X_{t_k} \in A_k)).$$

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The collection of all cylinder sets forms a π-system that generates $\mathcal{E}^*$ and thus $\mu^{\mathbf{X}}$ extends uniquely to a probability measure on $(E^*, \mathcal{E}^*)$. We call $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}})$ the *path space* of $\mathbf{X}$.

Symbolic dynamics of stochastic processes

Let $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}})$ be the path space of an (Ω, E) stochastic process $\mathbf{X}$. Define the shift map $s : E^* \to E^*$ by $s(x) = y$ where $y_i = x_{i+1}$, for each $i \in \mathbb{N}_0$. The dynamical system $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}}, s)$ is referred to as the symbolic dynamics of $\mathbf{X}$.

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- $\widehat{\mathcal{C}} := \{C\left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right)\}_{A \in \mathcal{C}} \in \mathcal{P}_{ar}(E^*)$ and
- the $(\Omega, \mathcal{C})$ stochastic process $\mathbf{X}_{\mathcal{C}} = (X_n^{\mathcal{C}})_{n=0}^{\infty}$ where, for each $n \in \mathbb{N}_0$ and $\omega \in \Omega$, $X_n^{\mathcal{C}}(\omega) = A$ whenever $X_n(\omega) \in A$.

Symbolic dynamics of a stochastic process

Proposition 10

Let $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}}, s)$ be the symbolic dynamics of an (Ω, E) stochastic process $\mathbf{X}$. Then for each $\mathcal{C} \in \mathcal{P}_{ar}(E)$, $H(\mathbf{X}_{\mathcal{C}}) = h^{KS}(s, \widehat{\mathcal{C}})$. In particular, whenever E is a discrete space, $H(\mathbf{X}) = h^{KS}(s, \widehat{\mathcal{A}})$, where $\mathcal{A}$ is the atomic partition of E.

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Let $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}}, s)$ *be the symbolic dynamics of an* (Ω, E) *stochastic process* $\mathbf{X}$*. Then for each* $\mathcal{C} \in \mathcal{P}_{ar}(E)$*,* $H(\mathbf{X}_{\mathcal{C}}) = h^{KS}(s, \widehat{\mathcal{C}})$*. In particular, whenever* E *is a discrete space,* $H(\mathbf{X}) = h^{KS}(s, \widehat{\mathcal{A}})$*, where* $\mathcal{A}$ *is the atomic partition of* E*.*

Corollary 11

Let $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}}, s)$ *be the symbolic dynamics of a discrete* (Ω, E) *stochastic process* $\mathbf{X}$ *and let* $\mathcal{A}$ *be the atomic partition of* E*. Then* $H(\mathbf{X}) = h^{KS}(s) = h^{KS}(s, \widehat{\mathcal{A}})$*, whenever* X_0 *has finite entropy.*

Symbolic dynamics of a dynamical system

Similarly, we can define the path space, $(\Omega^*, \Sigma^*, \mu^{(f,\mu)})$, for a dynamical system (Ω, Σ, μ, f) by setting

$$\mu^{(f,\mu)}(C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ t_0 & \cdots & t_n \end{smallmatrix}\right)) = \mu(\cap_{k=0}^{n}(f^{-t_k}(A_k))$$

on each cylinder set.

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For each $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, define the $(\Omega^*, \mathcal{C})$ stochastic process, $\mathbf{X}_{\mathcal{C}}^{(f,\mu)} = (X_n^{(f,\mu,\mathcal{C})})_{n=0}^{\infty}$, by $X_n^{(f,\mu,\mathcal{C})}(x) = A$, whenever $x_n \in A \in \mathcal{C}$, for all $n \in \mathbb{N}_0$ and $x \in \Omega^*$.

Proposition 12

For each $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, $H(\mathbf{X}_{\mathcal{C}}^{(f,\mu)}) = h^{KS}(f, \mathcal{C}) = h^{KS}(s, \widehat{\mathcal{C}})$. In particular, whenever Ω is a discrete space, $H(\mathbf{X}^{(f,\mu)}) = h^{KS}(f, \mathcal{A}) = h^{KS}(s, \widehat{\mathcal{A}})$, where $\mathcal{A}$ is the atomic partition of Ω.

Symbolic dynamics of a dynamical system

Corollary 13

Let $(\Omega^*, \mathcal{P}(\Omega)^*, \mu^{(f,\mu)}, s)$ *be the symbolic dynamics for the discrete dynamical system* $(\Omega, \mathcal{P}(\Omega), \mu, f)$ *and let* $\mathcal{A}$ *be the atomic partition of* Ω*. Then* $H(\mathbf{X}^{(f,\mu)}) = h^{KS}(s) = h^{KS}(f) = h^{KS}(f, \mathcal{A})$*, whenever* $\mathcal{A}$ *has finite entropy.*

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Theorem 14

If $\mathbf{X}$ *is a stationary* (Ω, E) *stochastic process then* $(E^*, \mathcal{E}^*, \mu^{\mathbf{X}}, s)$ *is a stationary dynamical system. Conversely, if* (Ω, Σ, μ, f) *is a stationary dynamical system, then* $\mathbf{X}^{(f,\mu)}$ *is a stationary stochastic process.*

Two fundamental properties of KS entropy

Property 1 (Zero on finite systems)

Let (Ω, Σ, μ, f) *be a dynamical system such that* $|\Omega| = N < \infty$*. Then* $h^{KS}(f) = 0$*.*

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For any dynamical system (Ω, Σ, μ, f)*, we have*

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Two fundamental properties of KS entropy

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Let (Ω, Σ, μ, f) *be a dynamical system such that* $|\Omega| = N < \infty$. *Then* $h^{KS}(f) = 0$.

Property 2 (Linear in time)

For any dynamical system (Ω, Σ, μ, f), *we have*

$$h^{KS}(f^n) = nh^{KS}(f), \text{ for all } n \in \mathbb{N}_0.$$

Let P be the transition matrix governing the unbiased random walk on the cycle. Then $H(P) = \ln 2$ and $H(P^2) = \frac{3}{2}\ln 2$. I.e. we have an example of a stochastic process whose entropy rate is nonzero on a finite system and is nonlinear in time.

Measurements

A *state space* is defined as a pair (X, K), where

(i) X is a real Banach space with norm $\|\cdot\|$,

(ii) K is a closed cone in X,

(iii) if $u, v \in K$, then $\|u\| + \|v\| = \|u + v\|$, and

(iv) if $u \in X$ and $\epsilon > 0$, then there exists $u_1, u_2 \in K$ such that $u = u_1 - u_2$ and $\|u_1\| + \|u_2\| < \|u\| + \epsilon$.

For any state space (X, K), there exists a unique positive linear functional $\tau : X \to \mathbb{R}$ such that $\tau(u) \leqslant \|u\|$, for $u \in X$, with equality when $u \in K$. We say that $u \in K$ is a *state* if $\tau(u) = 1$.

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Instruments

An *operation* is a positive, bounded linear operator $T : X \to X$, such that $0 \leqslant \tau(Tu) \leqslant \tau(u)$ for every $u \in K$. We denote by $\mathcal{O} := \mathcal{O}(X)$ the set of all operations on X. Finally, we define an *instrument* as a map $\mathcal{T} : \Sigma \to \mathcal{O}$ such that $\tau(\mathcal{T}(\Omega)u) = \tau(u)$, for all $u \in K$, and $\mathcal{T}(\cup_n E_n) = \sum_n \mathcal{T}(E_n)$, for any disjoint sequence of sets $\{E_n\} \subseteq \Sigma$, where convergence of the sum is in the strong operator topology.

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Classical mechanics

- Let Ω be a locally compact Hausdorff space and $\mathcal{B}$ be the Borel σ-algebra of Ω and take $(\Omega, \mathcal{B})$ to be the phase space.
- Let X be the real Banach space of all countably additive, regular, real-valued Borel measures on Ω equipped with the total variation norm
- Let K be the closed cone of X containing the nonnegative measures on Ω and set (X, K) to be the state space.
- The linear functional τ is given by $\tau(\nu) = \int_\Omega d\nu = \nu(\Omega)$ for any $\nu \in X$.
- We define the (classical) sharp measurement instrument $\mathcal{T}$ by

$$\mathcal{T}(E)\nu(A) = \nu(A \cap E) \quad \text{for } \nu \in S \text{ and } A, E \in \mathcal{B}.$$

Hilbert space quantum mechanics

Setting

- Let H be a Hilbert space.
- Let $X = S_1^{sa}(H)$ be the real Banach space of self-adjoint, trace class operators on H equipped with the trace class norm.
- Let $K = S_1^{+}(H)$ be the closed cone of X containing the positive, trace class operators on H and set (X, K) to be the state space.
- Let $(\Omega, \mathcal{P}(\Omega))$ be a discrete phase space.
- The linear functional τ is given by the trace, tr.

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- The linear functional τ is given by the trace, tr.

Given a collection of bounded operators, $(B_i)_{i\in\Omega} \subseteq B(H)$, such that $\sum_{i\in\Omega} B_i^* B_i = \mathbb{1}_H$ we define the instrument $\mathcal{T} : \Omega \to (X \to X)$ by

$$\mathcal{T}(E)\rho = \sum_{i\in E} B_i \rho B_i^* \quad \text{for each } \rho \in S \text{ and } E \in \Sigma,$$

where the sums are taken with respect to the strong operator topology if Ω is countably infinite.

Lüders-von Neumann instruments

Let $(P_i)_{i\in\Omega} \subset B(H)$ be a family of pairwise orthogonal projections such that $\sum_{i\in\Omega} P_i = \mathbb{1}$. The *Lüders-von Neumann instrument* generated by $(P_i)_{i\in\Omega}$, $\mathcal{T}$, is given by

$$\mathcal{T}(E)\rho = \sum_{i\in E} P_i \rho P_i \quad \text{for } \rho \in X \text{ and } E \in \Sigma.$$

Notice that $\mathcal{T}$ is defined by the collapse of wave function formula.

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Notice that $\mathcal{T}$ is defined by the collapse of wave function formula. Whenever the each projection, P_i, is rank-1, for all $i \in \Omega$, the Lüders-von Neumann instrument, $\mathcal{T}$, is called a *coherent states instrument.*

Probability measures on the path space

Let (Ω, Σ) be a phase space, (X, K) be a phase space, Θ a τ-preserving automorphism of X, $\mathcal{T}$ an instrument and $u \in K$ be a state. We will refer to $(\Theta, \mathcal{T}, u)$ as a *quantum stochastic process*.

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$$\widehat{\mu}^{(\Theta,\mathcal{T},u)}\left(C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ t_0 & \cdots & t_n \end{smallmatrix}\right)\right) = \mathrm{tr}(\mathcal{T}(A_n) \circ \Theta^{t_n - t_{n-1}} \circ \cdots \circ \Theta^{t_1 - t_0} \circ \mathcal{T}(A_0) \circ \Theta^{t_0} u).$$

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Remark

Unfortunately, unlike the measures $\mu^{(f,\mu)}$ or $\mu^{\mathbf{X}}$ in the path space of a dynamical system or a stochastic process, respectively, $\widehat{\mu}^{(\Theta,\mathcal{T},u)}$ is not well defined, in general. To create a well defined measure on (Ω^*, Σ^*), we will fix a sequence of times at which the system is to be measured.

Probability measures with a simple time sequence

For ease, we set $t_n = n$ for all $n \in \mathbb{N}_0$. Then, on each cylinder set $C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ t_0 & \cdots & t_n \end{smallmatrix}\right) \subseteq \Omega^*$, we define a measure, $\mu^{(\Theta,\mathcal{T},u)}$, by

$$\mu^{(\Theta,\mathcal{T},u)}\left(C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ 0 & \cdots & n \end{smallmatrix}\right)\right) := \tau(\mathcal{T}(A_n) \circ \Theta \circ \cdots \circ \Theta \circ \mathcal{T}(A_0)u).$$

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For $A_0, A_2 \in \Sigma$, we have

$$\begin{aligned} \mu^{(\Theta,\mathcal{T},u)}\left(C\left(\begin{smallmatrix} A_0 & A_2 \\ 0 & 2 \end{smallmatrix}\right)\right) &= \mu^{(\Theta,\mathcal{T},u)}\left(C\left(\begin{smallmatrix} A_0 & \Omega & A_2 \\ 0 & 1 & 2 \end{smallmatrix}\right)\right) \\ &= \tau(\mathcal{T}(A_2) \circ \Theta \circ \mathcal{T}(\Omega) \circ \Theta \circ \mathcal{T}(A_0)u), \end{aligned}$$

which is not necessarily equal to $\mathrm{tr}(\mathcal{T}(A_2) \circ \Theta^2 \circ \mathcal{T}(A_0)u)$.

Probability measures with a simple time sequence

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which is not necessarily equal to $\mathrm{tr}(\mathcal{T}(A_2) \circ \Theta^2 \circ \mathcal{T}(A_0)u)$. Therefore $\mu^{(\Theta,\mathcal{T},u)}(C\left(\begin{smallmatrix} A_0 & A_2 \\ 0 & 2 \end{smallmatrix}\right))$ is interpreted as the probability that a system in initial state u will be measured at times $0, 1, 2$ and record the measurement sequence (A_0, A_2) at times 0 and 2.

Slomczynski-Zyczkowski quantum entropy

The *Slomczynski-Zyczkowski (SZ) entropy* of $(\Theta, \mathcal{T}, u)$ with respect to $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$ is given by

$$h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{A_k \in \mathcal{C} \\ 0 \leqslant k \leqslant n-1}} \eta(\mu^{(\Theta,\mathcal{T},\rho)}(C \begin{pmatrix} A_0 & \cdots & A_{n-1} \\ 0 & \cdots & n-1 \end{pmatrix})),$$

whenever the limit exists.

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whenever the limit exists.

Lemma 15

Let everything be as above and let s be the shift map on $(\Omega^*, \Sigma^*, \mu^{(\Theta,\mathcal{T},u)})$. For each $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, define the $(\Omega^*, \mathcal{C})$ stochastic process $\mathbf{X}_{\mathcal{C}}^{(\Theta,\mathcal{T},u)}$ analogously to $\mathbf{X}_{\mathcal{C}}^{(f,\mu)}$ for classical dynamical systems. Then

$$H(\mathbf{X}_{\mathcal{C}}^{(\Theta,\mathcal{T},u)}) = h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}) = h^{KS}(s, \mu^{(\Theta,\mathcal{T},u)}, \widehat{\mathcal{C}}).$$

Two causes of randomness

The SZ measurement entropy of $(\Theta, \mathcal{T}, u)$ with respect to $\mathcal{C}$, denoted by $h^{SZ}_{\text{meas}}(\Theta, \mathcal{T}, \rho, \mathcal{C})$, quantifies the amount of randomness we observe in our system due to our choice of instrument, $\mathcal{T}$. It is given by

$$h^{SZ}_{\text{meas}}(\Theta, \mathcal{T}, u, \mathcal{C}) = h^{SZ}(\mathbb{1}, \mathcal{T}, u, \mathcal{C}).$$

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The remainder

$$h^{SZ}_{\text{dyn}}(\Theta, \mathcal{T}, u, \mathcal{C}) = h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}) - h^{SZ}_{\text{meas}}(\Theta, \mathcal{T}, u, \mathcal{C})$$

is referred to as the SZ dynamical entropy and quantifies the amount of randomness we observe in our system due to the dynamics, Θ.

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is referred to as the SZ dynamical entropy and quantifies the amount of randomness we observe in our system due to the dynamics, Θ. Luckily, for Lüders-von Neumann instruments and classical sharp measurements, $h^{SZ}_{\text{meas}}(\Theta, \mathcal{T}, u, \mathcal{C}) = 0$ and so

$$h^{SZ}_{\text{dyn}}(\Theta, \mathcal{T}, u, \mathcal{C}) = h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}),$$

so long as $H(\widehat{\mathcal{C}}) < \infty$.

SZ entropy with classical sharp instruments

Proposition 16

Let $(\Omega, \mathcal{B})$, (X, K), τ and $\mathcal{T}$ be as in the classical mechanics example. Let $\mu \in K$ be a state; i.e. a probability measure on $(\Omega, \mathcal{B})$, and $f : X \to X$ a measurable map so that $(\Omega, \mathcal{B}, \mu, f)$ is a DS. Let $T_f : X \to X$ be the automorphism known as the Koopman operator defined by

$$T_f(\nu)(A) := \nu(f^{-1}(A)) \text{ for all } \nu \in X \text{ and } A \in \mathcal{B}.$$

Then for each $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, $h^{KS}(f, \mu, \mathcal{C}) = h^{SZ}(T_f, \mathcal{T}, \mu, \mathcal{C})$.

SZ entropy with coherent states instruments

Let $\mathcal{T}$ be a coherent states instrument given by a family of orthogonal, rank-1 projections $(P_i)_{i \in \Omega}$ such that $P_i = |a_i\rangle\langle a_i|$ for each $i \in \Omega$.

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$$\mu^{(\Theta,\mathcal{T},\rho)}\left(C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ 0 & \cdots & n \end{smallmatrix}\right)\right) = \operatorname{tr}(\mathcal{T}(A_n)\circ\Theta\circ\cdots\circ\Theta\circ\mathcal{T}(A_0)\rho)$$

$$= \sum_{\substack{a_k\in A_k \\ 0\leqslant k\leqslant n}} \langle a_0|\rho|a_0\rangle \prod_{k=1}^{n} |\langle a_k|U|a_{k-1}\rangle|^2.$$

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$$= \sum_{\substack{a_k \in A_k \\ 0 \leqslant k \leqslant n}} \langle a_0|\rho|a_0\rangle \prod_{k=1}^{n} |\langle a_k|U|a_{k-1}\rangle|^2.$$

Thus $\mathbf{X}_{\mathcal{A}}^{(\Theta,\mathcal{T},\rho)}$ is a Markov process governed by the transition matrix $P = [|\langle a_i|U|a_j\rangle|^2]_{i,j\in\Omega}$ and initial distribution $p_{X_0^{(\Theta,\mathcal{T},\rho,\mathcal{A})}} = [\langle a_i|\rho|a_i\rangle]_{i\in\Omega}$.

SZ entropy is nonlinear in time

Theorem 17

Let $(\Omega, \mathcal{P}(\Omega))$ be a discrete phase space with $|\Omega| = N$ for some $N \in \mathbb{N}$, $\mathcal{T}$ a Lüders-von Neumann instrument, Θ a unitary transformation and $\rho \in S_1^+(H)$ a state. Then $h_{dyn}^{SZ}(\Theta^n, \mathcal{T}, \rho) \leqslant N$ for all $n \in \mathbb{N}$. Therefore, if $h_{dyn}^{SZ}(\Theta, \mathcal{T}, \rho) \neq 0$, then $h_{dyn}^{SZ}(\Theta^n, \mathcal{T}, \rho) \neq n h_{dyn}^{SZ}(\Theta, \mathcal{T}, \rho)$ for all sufficiently large $n \in \mathbb{N}$.

Hadamard walk

Let $H_C = \mathbb{C}^2$ with orthonormal basis $\{|R\rangle, |L\rangle\}$. Consider the vertex set $V = \mathbb{Z}$ or $\{0, \ldots, N-1\}$ for some $N \in \mathbb{N}$ with $N \geqslant 3$ and set $H_P = \ell_2(\mathbb{Z})$ or $\mathbb{C}^N$, respectively. Let $H = H_C \otimes H_P$. Define the integer shift operator, on H, by

$$S = \sum_{n=0}^{N-1} |R, n+1\rangle\langle R, n| + |L, n-1\rangle\langle L, n|,$$

where addition on the integers is done modulo N whenever $\Omega = \{0, \ldots, N-1\}$, and the unitary operator

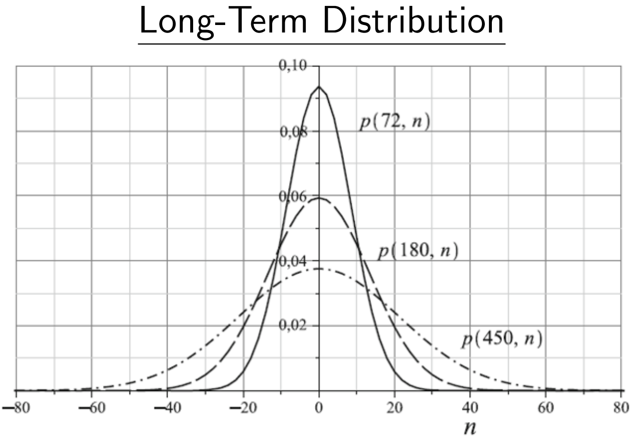
$$h := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

on H_C, referred to as the Hadamard matrix (or Hadamard coin/gate). The *Hadamard walk on V* is the unitary transformation, Θ, on X, given by

$$\Theta(\rho) = U\rho U^*, \text{ for all } \rho \in X, \text{ where } U = S(h \otimes \mathbb{1}_{H_P}).$$

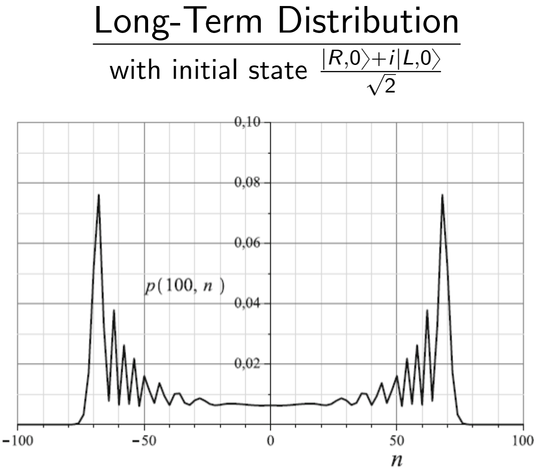
Hadamard walk with measurements

The Hadamard walk with measurements of the coin after each unit time produces the unbiased random walk.



Quantum Walks and Search Algorithms, R. Portugal

Hadamard walk without measurements



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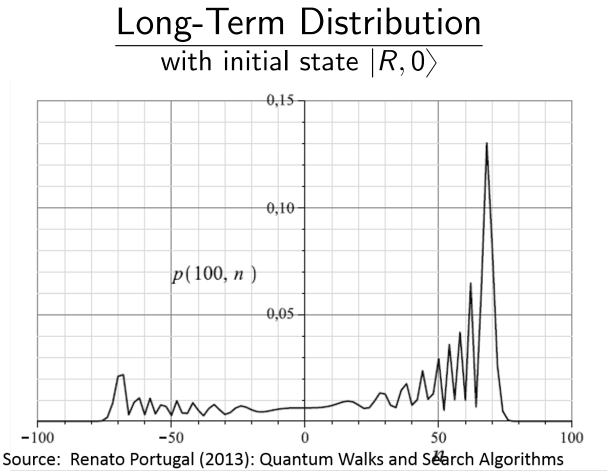

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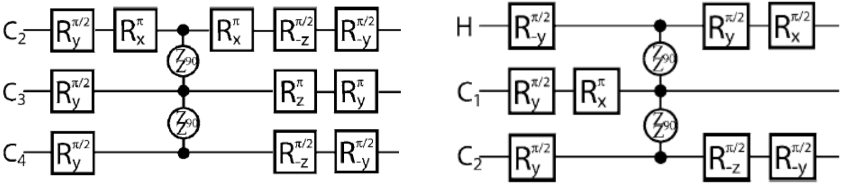
Implementation with Nuclear Magnetic Resonance



Experimental implementation of a discrete-time quantum random walk on an NMR quantum-information processor, Ryan et. al.

Two interpretations for measuring position

One option is to take the phase space to be $(C \times V, \mathcal{P}(C \times V))$, the coherent states instrument $\mathcal{T}$ to be given by the family $(P_e)_{e \in C \times V}$, where $P_{c,v} = |c, v\rangle\langle c, v|$, and calculate the SZ entropy with respect to the partition

$$\mathcal{C}_V = \{C_v\}_{v \in V}, \text{ where } C_v := \{|R, v\rangle, |L, v\rangle\}, \text{ for each } v \in V.$$

Two interpretations for measuring position

One option is to take the phase space to be $(C \times V, \mathcal{P}(C \times V))$, the coherent states instrument $\mathcal{T}$ to be given by the family $(P_e)_{e \in C \times V}$, where $P_{c,v} = |c, v\rangle\langle c, v|$, and calculate the SZ entropy with respect to the partition

$$\mathcal{C}_V = \{C_v\}_{v \in V}, \text{ where } C_v := \{|R, v\rangle, |L, v\rangle\}, \text{ for each } v \in V.$$

On the other hand we could take the phase space to be $(V, \mathcal{P}(V))$, define the projections

$$P_v = \mathbb{1}_{H_C} \otimes |v\rangle\langle v|, \text{ for each } v \in V,$$

and calculate the SZ entropy of the Lüders-von Neumann instrument $\mathcal{V}$, governed by the family $(P_v)_{v \in V}$, with respect to the atomic partition of V.

Two different outcomes for measuring position

Theorem 18

Let Θ be the the Hadamard walk on $V = \{0, \ldots, N-1\}$ with $|V| = N \geqslant 3$. Let $\mathcal{T}$ be the coherent states instrument given by the family of orthogonal projections $(P_e)_{e \in C \times V}$, $\rho = \frac{\mathbb{1}}{2N}$ and $\mathcal{C}_V$ the partition given on the previous slide. Then $h^{SZ}(\Theta, \mathcal{T}, \rho, \mathcal{C}_V) = \ln 2$ and $h^{SZ}(\Theta^2, \mathcal{T}, \rho, \mathcal{C}_V) = \frac{3}{2} \ln 2$.

Two different outcomes for measuring position

Theorem 18

Let Θ be the the Hadamard walk on $V = \{0, \ldots, N-1\}$ with $|V| = N \geqslant 3$. Let $\mathcal{T}$ be the coherent states instrument given by the family of orthogonal projections $(P_e)_{e \in C \times V}$, $\rho = \frac{\mathbb{1}}{2N}$ and $\mathcal{C}_V$ the partition given on the previous slide. Then $h^{SZ}(\Theta, \mathcal{T}, \rho, \mathcal{C}_V) = \ln 2$ and $h^{SZ}(\Theta^2, \mathcal{T}, \rho, \mathcal{C}_V) = \frac{3}{2} \ln 2$.

Theorem 19

Let Θ be the the Hadamard walk on $V = \{0, \ldots, N-1\}$ with $|V| = N \geqslant 3$. Let $\mathcal{V}$ be the Lüders-von Neumann instrument given by the family of orthogonal rank-2 projections $(P_v)_{v \in V}$ defined on the previous slide, $\rho = \frac{\mathbb{1}}{2N}$ and $\mathcal{A}$ the atomic partition of V. Then $h^{SZ}(\Theta, \mathcal{V}, \rho, \mathcal{A}) = \ln 2$ and $h^{SZ}(\Theta^2, \mathcal{V}, \rho, \mathcal{A}) = \frac{4}{3} \ln 2$.

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