The Nonlinearity of Quantum Dynamical Entropy

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Classical Mechanics

- (Phase space) $(\Omega, \mathcal{B})$, where Ω is a locally compact Hausdorff space.
- (State space) (X, K), where X (K) consists of all countably additive, regular, real-valued (positive) Borel measures.
- (States) $\mu \in K$, where μ is a probability measure.
- (Sharp Instrument) $\mathcal{T} : \mathcal{B} \to \mathcal{O}$ given by
$$\mathcal{T}(E)\nu(A) = \nu(A \cap E)$$
for $\nu \in S$ and $A, E \in \mathcal{B}$.
- (Measurements) $\tau : X \to \mathbb{R}$ given by
$$\tau(\nu) = \int_\Omega d\nu = \nu(\Omega)$$
for $\nu \in X$.
- (Dynamics) $T_f : X \to X$ is the *Koopman operator* given by
$$T_f(\nu)(A) := \nu(f^{-1}(A))$$
for $f : \Omega \to \Omega$, $\nu \in X$ and $A \in \mathcal{B}$.

Kolmogorov-Sinai Entropy

A dynamical system is a quadruple $(\Omega, \mathcal{B}, \mu, f)$ where

- $(\Omega, \mathcal{B}, \mu)$ is a probability space.
- $f : \Omega \to \Omega$ is a measurable map which governs the dynamics.

The partition dependent KS entropy is given by
$$h^{KS}(f, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n} H(\vee_{k=0}^{n-1} f^{-k}(\mathcal{C})).$$
The partition independent KS entropy is given by
$$h^{KS}(f) = \sup_{\substack{\mathcal{C}\in\mathcal{P}_{ar}(\Omega)\\ H(\mathcal{C})<\infty}} h^{KS}(f, \mathcal{C}).$$

Comparison to SZ Entropy

Given a dynamical system $(\Omega, \mathcal{B}, \mu, f)$, a partition $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$, and the remaining terms as defined for classical mechanics, we have
$$h^{KS}(f, \mathcal{C}) = h^{SZ}(T_f, \mathcal{T}, \mu, \mathcal{C}).$$

Slomczynski-Zyczkowski Quantum Dynamical Entropy

Fix a phase space (Ω, Σ), a state space (X, K), an instrument $\mathcal{T} : \Omega \to \mathcal{O}$, a state $u \in K$, a measurement functional $\tau : \mathbf{X} \to \mathbb{R}$ and a τ-preserving map $\Theta : X \to X$. We can then define a probability measure, $\mu^{(\Theta,\mathcal{T},u)}$, on (Ω^*, Σ^*), where $\Omega^* := E^{\mathbb{N}_0}$ and $\Sigma^* := \sigma(\cup_{n=0}^{\infty} \Sigma^n)$, by
$$\mu^{(\Theta,\mathcal{T},u)}(C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ 0 & \cdots & n \end{smallmatrix}\right)) := \tau(\mathcal{T}(A_n) \circ \Theta \circ \cdots \circ \Theta \circ \mathcal{T}(A_0)u).$$
The partition dependent dynamical SZ entropy is given by
$$h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{A_k\in\mathcal{C}\\ 0\le k\le n-1}} \eta(\mu^{(\Theta,\mathcal{T},\rho)}(C\left(\begin{smallmatrix} A_0 & \cdots & A_{n-1} \\ 0 & \cdots & n-1 \end{smallmatrix}\right))).$$
The partition independent dynamical SZ entropy is given by
$$h^{SZ}(\Theta, \mathcal{T}, u) = \sup_{\substack{\mathcal{C}\in\mathcal{P}_{ar}(\Omega)\\ H(\mathcal{C})<\infty}} h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}).$$

Hilbert Space Mechanics

- (Phase space) $(\Omega, \mathcal{P}(\Omega))$, where Ω is a discrete space.
- (State space) (X, K), where X (K) consists of all self-adjoint (positive), trace class operators on $\mathcal{H}$.
- (States) $\rho \in K$ such that $\mathrm{tr}(\rho) = 1$.
- (LvN Instrument) $\mathcal{T} : \mathcal{P}(\Omega) \to \mathcal{O}$ given by
$$\mathcal{T}(E)\rho = \sum_{i\in E} P_i \rho P_i$$
for $\rho \in X$, $E \in \Sigma$ and $(P_i)_{i\in\Omega}$ a family of pairwise orthogonal projections summing to $\mathbb{1}$.
- (Measurements) $\tau : X \to \mathbb{R}$ is the trace tr.
- (Dynamics) $\Theta : X \to X$ is a unitary transformation given by
$$\Theta(\rho) = U\rho U^*$$
for $\rho \in X$.

SZ Entropy is Nonlinear

Let all terms be defined as for Hilbert space mechanics such that $|\Omega| = N$ for some $N \in \mathbb{N}$. If $h^{SZ}(\Theta, \mathcal{T}, \rho) \neq 0$, then
$$h^{SZ}(\Theta^n, \mathcal{T}, \rho) \neq n h^{SZ}_{\mathrm{dyn}}(\Theta, \mathcal{T}, \rho)$$
for all sufficiently large $n \in \mathbb{N}$. This is in contrast to KS entropy, which is linear in time.

Hadamard Walk

The Hadamard walk is the most well-studied coined unitary quantum random walk. It is given by a unitary transformation, Θ, on the tensored Hilbert space $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$, where $\mathcal{H}_C = \mathbb{C}^2$ and $\mathcal{H}_P = \mathbb{C}^N$ for some $N \in \mathbb{N}$. To measure the position of the Hadamard walk, we can either use a LvN instrument, $\mathcal{T}$, of rank-1 projections with respect to the partition
$$\mathcal{C}_V = \{\{|R, n\rangle, |L, n\rangle\}\}_{n=0}^{N-1}$$
or we can use a LvN instrument, $\mathcal{V}$, of rank-2 projections,
$$(P_n)_{n=0}^{N-1}, \quad \text{with } P_n = \mathbb{1}_{H_C} \otimes |n\rangle\langle n|$$
with respect to the atomic partition, $\mathcal{A}$.

Ambiguity in Measurement

Letting $\rho = \mathbb{1}_H / 2N$, we find that
$$h^{SZ}(\Theta^2, \mathcal{T}, \rho, \mathcal{C}_V) \neq h^{SZ}(\Theta^2, \mathcal{V}, \rho, \mathcal{A}),$$
reflecting the sensitivity of a quantum system to measurement.

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Connection to Entropy Rate

Let $\mathbf{X}^{(\Theta,\mathcal{T},u)} = (X_n^{(\Theta,\mathcal{T},u)})_{n=0}^{\infty}$ be the stochastic process, with $X_n^{(\Theta,\mathcal{T},u)} : \Omega^* \to \Omega$, given by
$$X_n^{(\Theta,\mathcal{T},u)}(x) = x_n$$
for all $x \in \Omega^*$ and $n \in \mathbb{N}_0$. Then $\mu^{(\Theta,\mathcal{T},\rho)}$ is simply the probability distribution function of $\mathbf{X}^{(\Theta,\mathcal{T},u)}$.

Discrete Phase Space

Whenever the phase space (Ω, Σ) is discrete, we have that
$$h^{SZ}(\Theta, \mathcal{T}, u) = H(\mathbf{X}^{(\Theta,\mathcal{T},u)}).$$

Fix a partition $\mathcal{C} \in \mathcal{P}_{ar}(\Omega)$. We then define the restricted stochastic process $\mathbf{X}_{\mathcal{C}}^{(\Theta,\mathcal{T},u)} = (X_n^{(\Theta,\mathcal{T},u,\mathcal{C})})_{n=0}^{\infty}$, with $X_n^{(\Theta,\mathcal{T},u,\mathcal{C})} : \Omega^* \to \mathcal{C}$, given by
$$X_n^{(\Theta,\mathcal{T},u,\mathcal{C})} = i_{\mathcal{C}} \circ X_n^{(\Theta,\mathcal{T},u)},$$
where $i_{\mathcal{C}} : \Omega \to \mathcal{C}$ is given by $i_{\mathcal{C}}(\omega) = A \in \mathcal{C}$ whenever $\omega \in A$. The following holds:
$$h^{SZ}(\Theta, \mathcal{T}, u, \mathcal{C}) = H(\mathbf{X}_{\mathcal{C}}^{(\Theta,\mathcal{T},u)}).$$

Coherent States Instruments

Whenever the family (P_i) consists of rank-1 projections, the process $\mathbf{X}^{(\Theta,\mathcal{T},\rho)}$ is a Markov process with transitions probabilities $p_{i,j} = |\langle a_i|U|a_j\rangle|^2$ and initial distribution $\mu(i) = \langle a_i|\rho|a_i\rangle$, where $P_i = |a_i\rangle\langle a_i|$, for $i \in \Omega$. Therefore
$$h^{SZ}(\Theta, \mathcal{T}, \rho) = \sum_{j\in\Omega} \mu(j) \sum_{i\in\Omega} \eta(p_{i,j}).$$

Undefined Terms

symbol	meaning
$\mathcal{B}$	Borel σ-algebra
$\mathcal{O}$	Operations
H	Entropy (rate)
$\mathcal{P}_{ar}(\Omega)$	Countable partitions
$\mathcal{P}(\Omega)$	Power set
$\mathcal{H}$	Hilbert space
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$C\left(\begin{smallmatrix} A_0 & \cdots & A_n \\ 0 & \cdots & n \end{smallmatrix}\right)$	Cylinder set
η	$\eta(x) = -x \log x$