Arithmetic Progressions in the Polygonal Numbers

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Definition: Triangular Number

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Consider the sequence $\{T(n)\}_{n=1}^{\infty}$.

Motivating Question

Do infinitely-long arithmetic progressions exist in the triangular numbers? 

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Do arbitrarily-long arithmetic progressions exist in the triangular numbers? 

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What can we say about arithmetic progressions in the triangular numbers? 

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Definition: Arithmetic Progression

An arithmetic progression (AP) with a common difference $d$ is a sequence of numbers, finite or infinite, such that the difference of any two consecutive terms is a constant $d$. For the purposes of this talk, we will assume that $d$ is a positive integer. We will also assume that our sequence has at least three terms.
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- $T(2) = 3$, $T(72) = 2628$, and $T(102) = 153$ form an arithmetic progression with common difference $d = 2625$
Four-Term AP’s in the Triangular Numbers

Theorem (Mordell 1969; Sierpiński 1964)

There cannot be four squares in arithmetic progression with common difference $d \neq 0$.

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Letting \(B = 2b + 1, A = 2a + 1,\) and \(N = 2n + 1,\) we have

\[ B^2 - 2A^2 = -N^2. \]
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Letting \( B = 2b + 1 \), \( A = 2a + 1 \), and \( N = 2n + 1 \), we have

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B^2 - 2A^2 = -N^2.
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Supposing that \( B = NX \) and \( A = NY \), we can reduce this to

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Then \( b = \frac{(2n+1)X-1}{2} \) and \( a = \frac{(2n+1)Y-1}{2} \) for any positive integer \( n \).
Some Notes on Pell Equations:

We reduced our problem to the equation $B^2 - 2A^2 = -N^2$. For any divisor $q_i$ of $N$, we can let $B = q_i X_i$, $A = q_i Y_i$, $Q_i = N/q_i$, and consider $X_i^2 - 2Y_i^2 = -Q_i^2$. If this equation has a relatively prime solution, we get infinitely many solutions. This allows us to find all three-term arithmetic progressions beginning with $T(n)$. 
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Polygonal Numbers

Definition: Polygonal Number

Let $s$ be a fixed integer with $s \geq 3$. For a natural number $n$, the $n$-th Polygonal Number $P_s(n)$ is the number of points that are needed to create a regular $s$-gon with each side being of length $n - 1$. This number is given by

$$P_s(n) = \frac{s^2 - 1}{2} n^2 - \frac{s^2 - 2}{2} n.$$

Examples:

$P_3(4) = 10$, $P_4(4) = 16$, $P_5(4) = 22$.

Examples of Polygonal Numbers for $s = 3, 4, 5$ and $n = 1, 2, 3, 4$. 

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Some Remarks about AP’s in the Polygonal Numbers

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Then $b = \frac{(6n-1)X+1}{6}$ and $a = \frac{(6n-1)Y+1}{6}$.
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Using $X = 7$, $Y = 5$, and $n = 1$, we have $b = 6$ and $a = \frac{26}{6}$. 

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A Variation on AP’s in the Polygonal Numbers

Let $T = \{ P_3(n) \}_{n=1}^{\infty}$ be the triangular numbers and $S = \{ P_4(n) \}_{n=0}^{\infty}$ be the square numbers. Take $P = T \cup S$.

What can be said about arithmetic progressions in $P$?

This is being investigated by Dr. Lenny Jones and Joshua Ide from Shippensburg University.

What if we took $S$ to be a finite subset of the natural numbers and constructed $P = \bigcup_{s \in S} \{ P_s(n) \}_{n=1}^{\infty}$? Can we say anything about arithmetic progressions in $P$?

We do note that $\bigcup_{s=3}^{\infty} \{ P_s(n) \}_{n=1}^{\infty} = \mathbb{N} \{ 2 \}$, so we do need this to be a finite union above.
Let \( T = \{ P_3(n) \}_{n=1}^{\infty} \) be the triangular numbers and \( S = \{ P_4(n) \}_{n=0}^{\infty} \) be the square numbers. Take \( P = T \cup S \).
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We do note that $\bigcup_{s=3}^\infty \{ P_s(n) \}^\infty_{n=1} = \mathbb{N} \setminus \{2\}$, so we do need this to be a finite union above.
Take $f(x) \in \mathbb{Z}[x]$ and consider the sequence $F = \{f(n)\}_{n=1}^{\infty}$.

What can we say about arithmetic progressions in $F$?

Example ($f(x) = x^3 - x$)

Let $f(x) = x^3 - x$. Then $f(1) = 0$, $f(4) = 60$, and $f(5) = 120$ form a three-term arithmetic progression with common difference $d = 60$.

Example ($f(x) = x^3$)

Let $f(x) = x^3$. Let $F = \{f(n)\}_{n=1}^{\infty}$. Finding a three-term arithmetic progression in $F$ amounts to solving the Diophantine equation $A^3 - 2B^3 = -C^3$ in positive integers $A < B < C$. This equation has no solution by a theorem of Mordell from 1969.
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Thank You!