SOLUTIONS FOR HOMEWORK SET #9 FOR MATH 552

Assignment: p. 77, #1d,3,4,7; p. 81, #1,2,5,7, p. 92 #1,2,3,5,6.

p. 77, Problem #7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all $z \in D$ then $f(z)$ must be constant throughout D.

Proof. $f(x+iy) = u(x,y) + iv(x,y)$. Since $f(z)$ is real-valued for all $z \in D$ we have that $v(x,y) = 0$ for all $x+iy \in D$. Since f is analytic in D we have that the Cauchy-Riemann equations $u_x = v_y = 0$ and $u_y = -v_x = 0$ hold throughout D. Suppose $x(t), y(t)$ are two linear functions of $t \in [a,b]$ and $x(t)+iy(t) \in D$ for all $t \in [a,b]$. Then

$$\begin{aligned}\frac{d}{dt}u(x(t),y(t)) &= u_x(x(t),y(t))x'(t) + u_y(x(t),y(t))y'(t)\\ &= 0\,x'(t) + 0\,y'(t) = 0, \qquad t \in [a,b].\end{aligned}$$

Therefore $u(x(a),y(a)) = u(x(b),y(b))$. Let $z_0 \in D$ be fixed and suppose $z \in D$ is any other point. Since D is a domain there is a polygonal path (a finite concatenation of straight line segments) starting at z_0 and ending at z which lies entirely within D. u takes on the same value on the endpoints of each straight line segment, so $f = u$ has the same value at z as at z_0. Hence f is a constant function. □

p. 81, Problem #1b. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic on $\mathbb{R}^2$ and find a harmonic conjugate $v(x,y)$ for it.

Solution. Define $f(z) = 2z - z^3$. This function is analytic on $\mathbb{C}$. $f(x+iy) = 2(x+iy) - (x+iy)^3 = 2x + 2iy - (x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3) = 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3)$. We have that the real part of $f(z)$ is $u(x,y)$, and hence it is harmonic on $\mathbb{R}^2$. Also $v(x,y) = 2y - 3x^2y + y^3$ is the harmonic conjugate of u.

p. 81, Problem #7. Let the function $f(x+iy) = u(x,y) + iv(x,y)$ be analytic in a domain D and consider the families of level curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where $c_1, c_2 \in \mathbb{R}$. Prove that these families are orthogonal at each point $z_0 \in D$ where $f'(z_0) \neq 0$.

Proof. Suppose $z_0 = x_0 + iy_0$ is a point in D and $c_1 = u(x_0, y_0)$ and $c_2 = v(x_0, y_0)$. Define $\mathbf{n}_1 = \begin{pmatrix} u_x(x_0, y_0) \\ u_y(x_0, y_0) \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} v_x(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix}$. Since f is analytic at z_0 and $f'(z_0) \neq 0$ (by assumption), and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = u_x(x_0, y_0) - iu_y(x_0, y_0) = v_y(x_0, y_0) + iv_x(x_0, y_0)$ (using the Cauchy-Riemann equations), we see that $\mathbf{n}_1 \neq \mathbf{0}$ and $\mathbf{n}_2 \neq \mathbf{0}$. We learned in Math 241 (Vector Calculus) that the vector $\mathbf{n}_1$ is orthogonal to the tangent line $\updownarrow_1$ to the level curve $u(x, y) = c_1$ at the point (x_0, y_0). Similarly the vector $\mathbf{n}_2$ is orthogonal to the tangent line $\updownarrow_2$ to the level curve $v(x, y) = c_2$ at the point (x_0, y_0). The lines $\updownarrow_1$ and $\updownarrow_2$ will be orthogonal if and only if the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal, i.e. $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. Computing we have

$$\begin{aligned} \mathbf{n}_1 \cdot \mathbf{n}_2 &= \begin{pmatrix} u_x(x_0, y_0) \\ u_y(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} v_x(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix} \\ &= u_x(x_0, y_0)v_x(x_0, y_0) + u_y(x_0, y_0)v_y(x_0, y_0) \\ &= -u_x(x_0, y_0)u_y(x_0, y_0) + u_y(x_0, y_0)u_x(x_0, y_0) = 0, \end{aligned}$$

by the Cauchy-Riemann equations $v_x = -u_y$ and $v_y = u_x$. Thus $\updownarrow_1$ and $\updownarrow_2$ are orthogonal at (x_0, y_0). □

p. 92, Problem #6. Show that $|e^{z^2}| \leq e^{|z|^2}$.

Proof. $e^{z^2} = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}e^{2ixy}$, so $|e^{z^2}| = e^{x^2-y^2}$. Since $e^{-y^2} \leq 1 \leq e^{y^2}$ and $e^{x^2} > 0$ we have $e^{x^2-y^2} = e^{x^2}e^{-y^2} \leq e^{x^2}e^{y^2} \leq e^{x^2+y^2} = e^{|x+iy|^2} = e^{|z|^2}$. Therefore $|e^{z^2}| \leq e^{|z|^2}$. □