SUMMARY OF SOME OF THE MATH 550 MATERIAL
CONCERNING EXAM 3

Some Definitions:

- A 0-dimensional manifold $P \subset \mathbb{R}^n$ is a finite collection of points from $\mathbb{R}^n$.
- A 1-dimensional manifold $C \subset \mathbb{R}^n$ is called a “curve”. It is required that for every $p \in C$ there is an open ball $B(p, \varepsilon) \subset \mathbb{R}^n$, centered at $p$ of radius $\varepsilon > 0$, and an open interval $(-a, a)$ (or a half-open interval $(-a, 0]$), and a smooth local parameterization $f$ mapping $(-a, a)$ (or $(-a, 0]$) into $\mathbb{R}^n$, with range $C \cap B(p, \varepsilon)$, such that $f(0) = p$ and $f'(0) \neq 0$.
- A 2-dimensional manifold $S \subset \mathbb{R}^n$ is called a “surface”. It is required that for every $p \in S$ there is an open ball $B(p, \varepsilon) \subset \mathbb{R}^n$, and an open square $(-a, a) \times (-a, a)$ (or a rectangle $(-a, 0] \times (-a, a)$), and a smooth local parameterization $f$ mapping $(-a, a) \times (-a, a)$ (or $(-a, 0] \times (-a, a)$) onto $S \cap B(p, \varepsilon)$ such that $f(0, 0) = p$ and $Df(0, 0) : \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1.
- A 3-dimensional manifold $R \subset \mathbb{R}^n$ is called a “region”. It is required that for every $p \in S$ there is an open ball $B(p, \varepsilon) \subset \mathbb{R}^n$, and an open cube $(-a, a) \times (-a, a) \times (-a, a)$ (or a box $(-a, 0] \times (-a, a) \times (-a, a)$), and a smooth local parameterization $f$ mapping $(-a, a) \times (-a, a) \times (-a, a)$ (or $(-a, 0] \times (-a, a) \times (-a, a)$) onto $S \cap B(p, \varepsilon)$ such that $f(0, 0, 0) = p$ and $Df(0, 0, 0) : \mathbb{R}^3 \to \mathbb{R}^3$ is 1-1.

- $p$ is called an interior point of $X$ if the domain of the local parameterization $f$ is an open set. Otherwise $p$ is called a boundary point of $X$.
- If $X$ is a $k$-dimensional manifold, $k \geq 1$, and $p \in X$ then $T_pX = \{(p, v) \mid v = Df(0)u \in \mathbb{R}^n, u \in \mathbb{R}^k\}$, where $f$ is a local parameterization at $p$. If $k = 0$ then $T_pP = \{(p, 0) \mid 0 \in \mathbb{R}^n\}$.
- Remark: $(p, v) \in T_pX$ is interpreted as the vector $v \in \mathbb{R}^n$ tangent to $X$ at the point $p \in X$. $T_pX$ is a vector space with the operations $(p, v) + (p, w) = (p, v + w), (p, v)\alpha = (p, \alpha v)$, and zero vector $(p, 0)$. It is equipped with a dot product inherited from $\mathbb{R}^n$: $(p, v) \cdot (p, w) = v^T w$. We will often abuse notation slightly by saying that $v \in T_pX$ when in fact $(p, v) \in T_pX$.

- If $X$ is a $k$-dimensional manifold, $k \geq 0$, and $p \in X$, then define $\Omega_pX = \{(p, (e_1, \ldots, e_k)) \mid (e_1, \ldots, e_k)\}$ is an ordered orthonormal basis of $T_pX$. If $k = 0$ then $\Omega_pX = \{(p, ())\}$ is a set with one element. As with the tangent space we will often abuse notation by writing: $(e_1, \ldots, e_k) \in \Omega_pX$ instead of $(p, (e_1, \ldots, e_k)) \in \Omega_pX$.

- If $X$ is a $k$-dimensional manifold, $k \geq 0$, and $p \in X$, then an orientation of $T_pX$ is a mapping $\kappa_p : \Omega_pX \to \{+1, -1\}$ such that whenever $(e_1, \ldots, e_k) \in \Omega_pX$, and $A$ is a $k \times k$ matrix such that $(e_1, \ldots, e_k)A \in \Omega_pX$, then $\kappa_p((e_1, \ldots, e_k)A) = \kappa_p(e_1, \ldots, e_k)\det A$ (this condition is vacuous when $k = 0$).

- If $X$ is a $k$-dimensional manifold, $k \geq 0$, then an orientation of $X$ is a mapping $\kappa : \cup_{p \in X} \Omega_pX \to \{+1, -1\}$ such that $\kappa|_{\Omega_pX}$ is an orientation of $T_pX$ for each $p \in X$, and whenever $E_1, \ldots, E_k$ are continuous vector fields
on the connected open set \( U \subset \mathcal{X} \) such that \((E_1(p), \ldots, E_k(p)) \in \Omega_p \mathcal{X}\) for all \( p \in U \) then \( \kappa(p, (E_1(p), \ldots, E_k(p))) \) is a constant function of \( p \in U \). If \( k = 0 \) and \( \mathcal{X} = \mathcal{P} \) then \( \bigcup_{p \in \mathcal{P}} \Omega_p \mathcal{P} \) is in one-to-one correspondence with \( \mathcal{P} \), so an orientation of \( \mathcal{P} \) is any mapping \( \kappa: \mathcal{P} \to \{+1, -1\} \).

- Suppose \( \mathcal{X} \) is a \( k \)-dimensional manifold, \( k \geq 1 \), and \( \kappa: \bigcup_{p \in \mathcal{X}} \Omega_p \mathcal{X} \to \{+1, -1\} \) is an orientation of \( \mathcal{X} \). Suppose \( U \) is an open subset of \( \mathbb{R}^k \) and \( f: U \to \mathcal{X} \) is a parameterization of \( f(U) \subset \mathcal{X} \). Suppose that for all \( q \in U \) and for all \((e_1, \ldots, e_k) \in \Omega_f(q) \mathcal{X}\) we have that

\[
\kappa_{f(q)}(e_1, \ldots, e_k) \det([e_1, \ldots, e_k]^T Df(q)) > 0.
\]

Then we say that the parameterization \( f \) is consistent with the orientation \( \kappa \).

- Suppose \( \mathcal{X} \) is a \( k \)-dimensional manifold, \( k \geq 1 \), and \( \partial \mathcal{X} \) is the set of boundary points of \( \mathcal{X} \), which is a \((k - 1)\)-dimensional manifold. If \( p \in \partial \mathcal{X} \) let \( f \) denote a local parameterization of \( \mathcal{X} \) at \( p \). Then define \( u(p) \in T_p \mathcal{X} \) to be the unique unit vector such that \( u(p) \cdot v = 0 \) for all \( v \in T_p \partial \mathcal{X} \) and \( u(p) \cdot Df(0)e_1 > 0 \). \( u(p) \) is called the outward unit normal to \( \partial \mathcal{X} \) at \( p \).

- Suppose \( \mathcal{X} \) is a \( k \)-dimensional manifold, \( k \geq 1 \), and \( \partial \mathcal{X} \) is the set of boundary points of \( \mathcal{X} \). Suppose \( \kappa: \bigcup_{p \in \partial \mathcal{X}} \Omega_p \partial \mathcal{X} \to \{+1, -1\} \) is an orientation of \( \mathcal{X} \). Then the boundary orientation on \( \partial \mathcal{X} \) it induced by \( \kappa \) is the mapping \( \tilde{\kappa}: \bigcup_{p \in \partial \mathcal{X}} \Omega_p \partial \mathcal{X} \to \{+1, -1\} \) defined by

\[
\tilde{\kappa}_p(e_1, \ldots, e_{k-1}) = \kappa_p(u(p), e_1, \ldots, e_{k-1})
\]

for all \( p \in \partial \mathcal{X} \) and \((e_1, \ldots, e_{k-1}) \in \Omega_p \partial \mathcal{X} \), where \( u(p) \) is the outward unit normal to \( \partial \mathcal{X} \) at \( p \).

- A 0-form on \( \mathbb{R}^n \) is an ordinary smooth real-valued function \( f: U \to \mathbb{R} \), defined on an open subset \( U \subset \mathbb{R}^n \).

- If \( k \geq 1 \) and \( f_1, \ldots, f_k \) are 0-forms defined on \( U \subset \mathbb{R}^n \) then define the basic \( k \)-form \( df_1 \wedge df_2 \wedge \cdots \wedge df_k \) to be the following mapping:

\[
(df_1 \wedge df_2 \wedge \cdots \wedge df_k)_x(v_1, \ldots, v_k) = \det \begin{pmatrix}
Df_1(x)v_1 & \cdots & Df_1(x)v_k \\
Df_2(x)v_1 & \cdots & Df_2(x)v_k \\
\vdots & \ddots & \vdots \\
Df_k(x)v_1 & \cdots & Df_k(x)v_k 
\end{pmatrix},
\]

where \( x \in U \) and \( v_1, \ldots, v_k \in \mathbb{R}^n \).

- If \( k \geq 1 \) and \( g_1, \ldots, g_m \) are 0-forms defined on \( U \subset \mathbb{R}^n \), and \( \omega^1, \ldots, \omega^m \) are basic \( k \)-forms on \( U \), then \( g_1 \omega^1 + \cdots + g_m \omega^m \) is called a \( k \)-form on \( U \). It is a mapping of the arguments \( x \in U \) and \( v_1, \ldots, v_k \in \mathbb{R}^n \) such that

\[
(g_1 \omega^1 + \cdots + g_m \omega^m)_x(v_1, \ldots, v_k) = g_1(x)\omega^1_x(v_1, \ldots, v_k) + \cdots + g_m(x)\omega^m_x(v_1, \ldots, v_k).
\]

- If \( \mathcal{P} \) is a 0-dimensional manifold in \( \mathbb{R}^n \), \( \kappa: \mathcal{P} \to \{+1, -1\} \) is an orientation of \( \mathcal{P} \), and \( f \) is a 0-form defined on the open set \( U \subset \mathbb{R}^n \), where \( \mathcal{P} \subset U \), then define \( \int_{\mathcal{P}, n} f = \sum_{p \in \mathcal{P}} f(p) \kappa(p) \).

- Suppose \( \mathcal{X} \) is a \( k \)-dimensional manifold, \( k \geq 1 \), and \( \kappa: \bigcup_{p \in \mathcal{X}} \Omega_p \mathcal{X} \to \{+1, -1\} \) is an orientation of \( \mathcal{X} \). Suppose \( U \) is an open subset of \( \mathbb{R}^k \) and \( \rho: U \to \mathcal{X} \) is a parameterization of \( \rho(U) \subset \mathcal{X} \) which is consistent with the orientation \( \kappa \). Suppose \( \omega = g_1 \omega^1 + \cdots + g_m \omega^m \) is a \( k \)-form on an open
subset $V \subset \mathbb{R}^n$ where $\rho(U) \subset V$. Let $u = (u_1, \ldots, u_k)$ be the standard Cartesian coordinates on $\mathbb{R}^k$. Then define

$$
\int_{\rho(U) \cap \omega} \omega = \int_{u \in U} \omega_{\rho(u)}(D\rho(u)\hat{e}_1, \ldots, D\rho(u)\hat{e}_k) \, du_1 \ldots du_k.
$$

- If $f$ is a 0-form on $\mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ are the standard Cartesian coordinates on $\mathbb{R}^n$ then define $df$ to be the 1-form $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \, dx_i$. This agrees with our earlier definition of basic 1-forms. In fact, if $f(x) = x_i$ then $df_x(v) = v_i$ for all $x \in \mathbb{R}^n$; hence we write $df = dx_i$, omitting any reference to $x$.
- If $\omega = \sum_{j=1}^m g_j \omega^j$ is a $k$-form on $U \subset \mathbb{R}^n$, where the basic $k$-forms $\omega^j$ are wedge products of various $dx_i$'s, $i = 1, \ldots, n$, (and hence are independent of $x$) then define the $(k+1)$-form $d\omega$ by the rule $d\omega = \sum_{j=1}^m dg_j \wedge \omega^j$.

Some Important Facts:

- If $\omega$ is a $k$-form and $\eta$ is an $l$-form then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ is a $(k+l)$-form.
- If $\omega$, $\eta$, and $\zeta$ are differential forms then $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$.
- If $\omega$ is a $k$-form, $\eta$ and $\zeta$ are $l$-forms, and $f$ and $g$ are 0-forms, then $\omega \wedge (f \eta + g \zeta) = f \omega \wedge \eta + g \omega \wedge \zeta$.
- $d[M(x,y) \, dx + N(x,y) \, dy] = (N_x - M_y) \, dx \wedge dy$.
- $d[f(x,y,z) \, dx + g(x,y,z) \, dy + h(x,y,z) \, dz] = (h_y - g_z) \, dy \wedge dz + (f_z - h_x) \, dz \wedge dx + (g_z - f_y) \, dx \wedge dy$.
- If $F = (f, g, h)$ is a vector field then $\nabla \cdot F = f_x + g_y + h_z$.
- If $F = (f, g, h)$ is a vector field on $\mathbb{R}^3$, $\omega = f(x,y,z) \, dx + g(x,y,z) \, dy + h(x,y,z) \, dz$ is the corresponding 1-form, and $\rho: \mathbb{R} \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}$ we have $\omega_{\rho(u)}(D\rho(u)\hat{e}_1) = F(\rho(u)) \cdot D\rho(u)$.
- If $F = (f, g, h)$ is a vector field on $\mathbb{R}^3$, $\omega = f(x,y,z) \, dy \wedge dx + g(x,y,z) \, dz \wedge dx + h(x,y,z) \, dx \wedge dy$ is the corresponding 2-form, and $\rho: \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}^2$ we have $\omega_{\rho(u)}(D\rho(u)\hat{e}_1, D\rho(u)\hat{e}_2, D\rho(u)\hat{e}_3) = F(\rho(u)) \cdot [D\rho(u)\hat{e}_1 \times D\rho(u)\hat{e}_2]$.
- If $\omega = f(x,y,z) \, dx \wedge dy \wedge dz$ and $\rho: \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}^3$ we have $\omega_{\rho(u)}(D\rho(u)\hat{e}_1, D\rho(u)\hat{e}_2, D\rho(u)\hat{e}_3) = f(\rho(u)) \det[D\rho(u)]$.
- **Stokes’ Theorem** Suppose $\mathcal{X} \subset \mathbb{R}^n$ is a closed and bounded $k$-dimensional manifold, $k \geq 1$, with orientation $\kappa: \cup_{p \in \mathcal{X}} \Omega_p \mathcal{X} \to \{+1, -1\}$. Suppose $\partial \mathcal{X}$ is the set of boundary points of $\mathcal{X}$, and $\bar{\kappa}: \cup_{p \in \partial \mathcal{X}} \Omega_p \partial \mathcal{X} \to \{+1, -1\}$ is the induced boundary orientation. Suppose $\omega$ is a $(k-1)$-form on $\mathbb{R}^n$, and $d\omega$ is its exterior derivative (a $k$-form on $\mathbb{R}^n$). Then

$$
\int_{\mathcal{X} \cup \partial \mathcal{X}} d\omega = \int_{\partial \mathcal{X} \cup \bar{\kappa}} \omega.
$$

Study Questions:

(1) page 519, number 2, 3, 5.
(2) Suppose \( f(x, y) = \frac{3}{2}x^2 + \frac{7}{2}y^2 - 5xy \) and \( C^* \) is the oriented line segment from \((1, 3)\) to \((5, 2)\). Show that \( \int_{C^*} df = f(5, 2) - f(1, 3) \) by calculating both sides.

(3) Suppose \( \omega = -y dx + x dy \) is a 1-form on \( \mathbb{R}^3 \) corresponding to the vector field \( F = (-y, x, 0) \). Suppose \( S = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 + z^2 = 1 \} \) is the upper half of a unit sphere. Let \( C = \partial S \) be the unit circle in the \( xy \) plane. Suppose \( S^* \) is oriented such that the “positive side” of the surface is the one away from the origin. Let \( C^* \) have the induced boundary orientation. Show that \( \int_S \omega = \int_{C^*} \omega \) by calculating both sides.

(4) Suppose \( \omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \) is a 2-form on \( \mathbb{R}^3 \). Suppose \( \mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \} \) be the unit sphere together with its interior. Let \( S = \partial \mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \) be the unit sphere. Suppose \( \mathcal{R}^* \) has the standard orientation of \( \mathbb{R}^3 \) and \( S^* \) has the induced boundary orientation. Show that \( \int_{\mathcal{R}} \omega = \int_{S^*} \omega \) by calculating both sides.

(5) Suppose \( S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1 \} \) and define the parameterization \( \rho(u, v) = (1 - u - v, u, v) \), where \( (u, v) \in \mathbb{R}^2 \). Let \( \mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1) \), and define the orientation \( \kappa_p(e_1, e_2) = \mathbf{n} \cdot e_1 \times e_2 \), for all \( p \in S \) and all \( (e_1, e_2) \in \Omega_p S \). Show that the parameterization \( \rho \) is consistent with the orientation \( \kappa \) on \( S \).

(6) Suppose \( S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1, z \geq 0 \} \) and \( C = \{(x, y, 0) \in \mathbb{R}^3 \mid x + y = 1 \} \). Suppose \( S \) is equipped with the orientation \( \kappa \) defined in problem (5). Show that \( \mathbf{u}(p) = \frac{1}{\sqrt{6}}(1, 1, -2) \) is the outward unit normal to \( C \). If \( \mathbf{e}_1 \) is the induced boundary orientation on \( C \), and \( \mathbf{e}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0) \) is an orthonormal basis of \( T_p C \), then compute \( \kappa_p(e_1) \).

**Solutions to Some Problems:**

**Problem 3.** First we will compute \( \int_{C^*} \omega \), and as always we need a parameterization of \( C^* \) consistent with the orientation. \( C \) is the unit circle in the \( xy \) plane, so as discussed in class we parameterize it as follows:

\[
\rho(\theta) = (\cos \theta, \sin \theta, 0), \quad \theta \in [0, 2\pi].
\]

From this we calculate

\[
D\rho(\theta) = (-\sin \theta, \cos \theta, 0).
\]

Recall the ordered basis \((e_2, e_3, e_0)\) associated to each point in the coordinate domain of spherical coordinates. \((e_0, e_1)\) is a positively oriented orthonormal basis of the tangent space of the unit sphere, \( \mathbf{u}(p) = e_2 \) is the outward unit normal to \( C \) and \( e_0 \) is a basis vector of \( T_p C \). Hence \( \rho \), which parameterizes \( C \) via the coordinate \( \theta \) is consistent with the induced boundary orientation of \( C^* \).

The 1-form \( \omega = -y dx + x dy + 0 dz \) corresponds to the vector field \( F = (-y, x, 0) \). By one of our important facts,

\[
\omega_{\rho(\theta)}(D\rho(\theta)e_1) = F(\rho(\theta)) \cdot D\rho(\theta) = (-\sin \theta, \cos \theta, 0) \cdot (-\sin \theta, \cos \theta, 0) = \sin^2 \theta + \cos^2 \theta = 1.
\]

Now by the definition of the integral of a 1-form over an oriented 1-dimensional manifold we have

\[
\int_{C^*} \omega = \int_0^{2\pi} \omega_{\rho(\theta)}(D\rho(\theta)e_1) \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi.
\]
Now we must compute $\int_S d\omega$; we begin with a parameterization of $S^*$. Define $\rho(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle$ for $(\phi, \theta) \in U = (0, \pi/2) \times (0, 2\pi)$. $\rho(U)$ is almost all of $S$. As we proved in class this parameterization is consistent with the orientation of $S^*$. Computing we find

$$D\rho(\phi, \theta) = \begin{pmatrix} \cos \phi \cos \theta & -\sin \phi \sin \theta \\ \cos \phi \sin \theta & \sin \phi \cos \theta \\ -\sin \phi & 0 \end{pmatrix}. $$

The 2-form $d\omega = d(-y \, dx + x \, dy) = -dy \wedge dx + dx \wedge dy = 2 \, dx \wedge dy$ corresponds to the vector field $F(x, y, z) = \langle 0, 0, 2 \rangle$. By one of our important facts we have

$$(d\omega)_{\rho(\phi, \theta)}(D\rho(\phi, \theta) \hat{e}_1, D\rho(\phi, \theta) \hat{e}_2) = F(\rho(\phi, \theta)) \cdot D\rho(\phi, \theta) \hat{e}_1 \times D\rho(\phi, \theta) \hat{e}_2$$

$$= \det \begin{pmatrix} 0 & 0 & 2 \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{pmatrix}$$

$$= 2 \sin \phi \cos \phi. $$

Thus by the definition of the integral of a 2-form over an oriented 2-dimensional manifold we have

$$\int_{S^*} d\omega = \int_U (d\omega)_{\rho(\phi, \theta)}(D\rho(\phi, \theta) \hat{e}_1, D\rho(\phi, \theta) \hat{e}_2) \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 2 \sin \phi \cos \phi \, d\phi \, d\theta = \pi \int_0^{\pi/2} 2 \sin^2 \phi \, d\theta = 2\pi.$$

Thus we found that $\int_{C^*} \omega = 2\pi$ and $\int_{S^*} d\omega = 2\pi$, so we have verified that $\int_{S^*} d\omega = \int_{C^*} \omega$.

**Problem 4.** We begin with the calculation of $\int_{S^*} \omega$. Define the parameterization $\rho(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle$ for $(\phi, \theta) \in U = (0, \pi) \times (0, 2\pi)$. $\rho(U)$ is almost all of $S$. Recall the ordered basis $(e_v, e_\phi, e_\theta)$ associated to each point in the coordinate domain of spherical coordinates is positively oriented with respect to the standard orientation of $\mathbb{R}^3$. $u(p) = e_\phi$ is the unit outward normal to $S$. Hence $(e_\phi, e_\theta)$ is a positively oriented orthonormal basis of the tangent space of the unit sphere with respect to the induced boundary orientation. As we have seen in class this implies that the parameterization $\rho$ is consistent with the orientation of $S^*$.

The 2-form $\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$ is associated with the vector field $F(x, y, z) = \langle x, y, z \rangle$. By one of our important facts we have

$$\omega_{\rho(\phi, \theta)}(D\rho(\phi, \theta) \hat{e}_1, D\rho(\phi, \theta) \hat{e}_2) = F(\rho(\phi, \theta)) \cdot D\rho(\phi, \theta) \hat{e}_1 \times D\rho(\phi, \theta) \hat{e}_2$$

$$= \det \begin{pmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{pmatrix}$$

$$= \sin \phi. $$
(Obviously there was a bit of calculation skipped here.) Thus by the definition of the integral of a 2-form over an oriented 2-dimensional manifold we have

\[
\int_{S^*} \omega = \int_U \omega_{\rho(\phi, \theta)}(D\rho(\phi, \theta)\hat{e}_1, D\rho(\phi, \theta)\hat{e}_2) \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = -2\pi \cos \phi \bigg|_{\phi=0}^{\phi=\pi} = 4\pi.
\]

One of our important facts allows us to compute \( d\omega = 3 \, dx \wedge dy \wedge dz \), since the volume of the unit sphere \( \mathcal{R} \) is \( \frac{4}{3} \pi \). Thus we have computed both integrals and obtained the same answer.

**Problem 5.** To verify consistency we must check the condition

\[
\kappa_{\rho(u,v)}(e_1, e_2) \det[(e_1, e_2)^T D\rho(u,v)] > 0
\]

for all \((e_1, e_2) \in \Omega_{\rho(u,v)}\mathcal{S}\). First of all

\[
D\rho(u,v) = \begin{pmatrix}
\frac{\partial(1-u-v)}{\partial(u)} & \frac{\partial(1-u-v)}{\partial(v)} \\
\frac{\partial u}{\partial(u)} & \frac{\partial u}{\partial(v)} \\
\frac{\partial v}{\partial(u)} & \frac{\partial v}{\partial(v)}
\end{pmatrix} = \begin{pmatrix}
-1 & -1 \\
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

Suppose \( e_1 = \langle a_1, b_1, c_1 \rangle \) and \( e_2 = \langle a_2, b_2, c_2 \rangle \). Then

\[
\det[(e_1, e_2)^T D\rho(u,v)] = \det \begin{pmatrix}
a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\
a_1 & b_1 & b_2 & c_1 & c_2 \\
1 & 0 & 1
\end{pmatrix} = \det \begin{pmatrix}
a_1 & b_1 & c_1 & -1 & -1 \\
a_2 & b_2 & c_2 & 1 & 0 \\
0 & 1
\end{pmatrix} = (-a_1 + b_1)(-a_2 + c_2) - (-a_2 + b_2)(-a_1 + c_1) = a_2c_1 - a_1c_2 + a_3b_1 - b_1c_1.
\]

On the other hand

\[
\kappa_{\rho(u,v)}(e_1, e_2) = \hat{u} \cdot e_1 \times e_2 = \det \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix} = \frac{1}{\sqrt{3}}[b_1c_2 - b_2c_1 - (a_1c_2 - a_2c_1) + a_1b_2 - a_2b_1]
\]

Thus

\[
\kappa_{\rho(u,v)}(e_1, e_2) \det[(e_1, e_2)^T D\rho(u,v)] = \frac{1}{\sqrt{3}}[b_1c_2 - b_2c_1 + a_2c_1 - a_1c_2 + a_1b_2 - a_2b_1]^2 > 0
\]

as we needed to show.

**Problem 6.** To check that \( u(p) \) is the outward unit normal first note that \( \hat{u} \cdot u(p) = \frac{1}{\sqrt{3}}(1,1,1) \cdot \frac{1}{\sqrt{6}}(1,1,-2) = \frac{1}{\sqrt{18}}(1(1) + 1(1) + 1(-2)) = 0 \), so \( u(p) \in T_p\mathcal{S} \). Two points on the boundary line \( \mathcal{C} \) are \( (1,0,0) \) and \( (0,1,0) \), so a tangent vector
to this line is $(0,1,0) - (1,0,0) = (−1,1,0)$. Also $\mathbf{u}(p) \cdot (−1,1,0) = \frac{1}{\sqrt{3}}(1,1,1) \cdot (−1,1,0) = 0$, so $\mathbf{u}(p)$ is perpendicular to $T_p\mathcal{C}$. Also since $\mathcal{S}$ has $z \geq 0$ whereas the $z$-component of $\mathbf{u}(p)$ is negative, we have that $\mathbf{u}(p)$ points “outward” from $\mathcal{S}$.

To compute $\kappa_p(\mathbf{e}_1)$ we use the definition of the induced boundary orientation:

$$\kappa_p(\mathbf{e}_1) = \kappa_p(\mathbf{u}(p), \mathbf{e}_1) = \mathbf{n} \cdot \mathbf{u}(p) \times \mathbf{e}_1 = \mathbf{n} \times \mathbf{u}(p) \cdot \mathbf{e}_1$$

$$= \frac{1}{\sqrt{3}}(1,1,1) \times \frac{1}{\sqrt{6}}(1,1,-2) \cdot \mathbf{e}_1 = \frac{1}{\sqrt{3}}(-3,3,0) \cdot \mathbf{e}_1$$

$$= \frac{1}{\sqrt{2}}(−1,1,0) \cdot \mathbf{e}_1$$

This dot product is either $+1$ or $−1$ according as $\mathbf{e}_1 = \frac{1}{\sqrt{3}}(−1,1,0)$. So the positive direction on $\mathcal{C}$ is in the direction of $\frac{1}{\sqrt{2}}(−1,1,0)$. 
