SUMMARY OF SOME OF THE MATH 550 MATERIAL CONCERNING EXAM 3

Some Definitions:

- A 0-dimensional manifold $\mathcal{P} \subset \mathbb{R}^n$ is a finite collection of points from $\mathbb{R}^n$.
- A 1-dimensional manifold $\mathcal{C} \subset \mathbb{R}^n$ is called a "curve". It is required that for every $p \in \mathcal{C}$ there is an open ball $B(p, \epsilon) \subset \mathbb{R}^n$, centered at p of radius $\epsilon > 0$, and an open interval $(-a, a)$ (or a half-open interval $(-a, 0]$), and a smooth local parameterization f mapping $(-a, a)$ (or $(-a, 0]$) into $\mathbb{R}^n$, with range $\mathcal{C} \cap B(p, \epsilon)$, such that $f(0) = p$ and $f'(0) \neq \mathbf{0}$.
- A 2-dimensional manifold $\mathcal{S} \subset \mathbb{R}^n$ is called a "surface". It is required that for every $p \in \mathcal{S}$ there is an open ball $B(p, \epsilon) \subset \mathbb{R}^n$, and an open square $(-a, a) \times (-a, a)$ (or a rectangle $(-a, 0] \times (-a, a)$), and a smooth local parameterization f mapping $(-a, a) \times (-a, a)$ (or $(-a, 0] \times (-a, a)$) onto $\mathcal{S} \cap B(p, \epsilon)$ such that $f(0, 0) = p$ and $Df(0, 0) \colon \mathbb{R}^2 \to \mathbb{R}^n$ is 1-1.
- A 3-dimensional manifold $\mathcal{R} \subset \mathbb{R}^n$ is called a "region". It is required that for every $p \in \mathcal{S}$ there is an open ball $B(p, \epsilon) \subset \mathbb{R}^n$, and an open cube $(-a, a) \times (-a, a) \times (-a, a)$ (or a box $(-a, 0] \times (-a, a) \times (-a, a)$), and a smooth local parameterization f mapping $(-a, a) \times (-a, a) \times (-a, a)$ (or $(-a, 0] \times (-a, a) \times (-a, a)$) onto $\mathcal{S} \cap B(p, \epsilon)$ such that $f(0, 0, 0) = p$ and $Df(0, 0, 0) \colon \mathbb{R}^3 \to \mathbb{R}^n$ is 1-1.
- p is called an *interior* point of $\mathcal{X}$ if the domain of the local parameterization f is an open set. Otherwise p is called a *boundary* point of $\mathcal{X}$.
- If $\mathcal{X}$ is a k-dimensional manifold, $k \geq 1$, and $p \in \mathcal{X}$ then $T_p\mathcal{X} = \{(p, \mathbf{v}) \mid \mathbf{v} = Df(\mathbf{0})\mathbf{u} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^k\}$, where f is a local parameterization at p. If $k = 0$ then $T_p\mathcal{P} = \{(p, \mathbf{0}) \mid \mathbf{0} \in \mathbb{R}^n\}$.
- *Remark:* $(p, \mathbf{v}) \in T_p\mathcal{X}$ is interpreted as the vector $\mathbf{v} \in \mathbb{R}^n$ tangent to $\mathcal{X}$ at the point $p \in \mathcal{X}$. $T_p\mathcal{X}$ is a vector space with the operations $(p, \mathbf{v}) + (p, \mathbf{w}) = (p, \mathbf{v} + \mathbf{w})$, $(p, \mathbf{v})\alpha = (p, \mathbf{v}\alpha)$, and zero vector $(p, \mathbf{0})$. It is equipped with a dot product inherited from $\mathbb{R}^n$: $(p, \mathbf{v}) \cdot (p, \mathbf{w}) = \mathbf{v}^T\mathbf{w}$. We will often abuse notation slightly by saying that $\mathbf{v} \in T_p\mathcal{X}$ when in fact $(p, \mathbf{v}) \in T_p\mathcal{X}$.
- If $\mathcal{X}$ is a k-dimensional manifold, $k \geq 0$, and $p \in \mathcal{X}$, then define $\Omega_p\mathcal{X} = \{(p, (\mathbf{e}_1, \ldots, \mathbf{e}_k)) \mid (\mathbf{e}_1, \ldots, \mathbf{e}_k)$ is an ordered orthonormal basis of $T_p\mathcal{X}\}$. If $k = 0$ then $\Omega_p\mathcal{X} = \{(p, ())\}$ is a set with one element. As with the tangent space we will often abuse notation by writing: $(\mathbf{e}_1, \ldots, \mathbf{e}_k) \in \Omega_p\mathcal{X}$ instead of $(p, (\mathbf{e}_1, \ldots, \mathbf{e}_k)) \in \Omega_p\mathcal{X}$.
- If $\mathcal{X}$ is a k-dimensional manifold, $k \geq 0$, and $p \in \mathcal{X}$, then an *orientation* of $T_p\mathcal{X}$ is a mapping $\kappa_p \colon \Omega_p\mathcal{X} \to \{+1, -1\}$ such that whenever $(\mathbf{e}_1, \ldots, \mathbf{e}_k) \in \Omega_p\mathcal{X}$, and A is a $k \times k$ matrix such that $(\mathbf{e}_1, \ldots, \mathbf{e}_k)A \in \Omega_p\mathcal{X}$, then $\kappa_p((\mathbf{e}_1, \ldots, \mathbf{e}_k)A) = \kappa_p(\mathbf{e}_1, \ldots, \mathbf{e}_k) \det A$ (this condition is vacuous when $k = 0$).
- If $\mathcal{X}$ is a k-dimensional manifold, $k \geq 0$, then an *orientation* of $\mathcal{X}$ is a mapping $\kappa \colon \cup_{p \in \mathcal{X}} \Omega_p\mathcal{X} \to \{+1, -1\}$ such that $\kappa|_{\Omega_p\mathcal{X}}$ is an orientation of $T_p\mathcal{X}$ for each $p \in \mathcal{X}$, and whenever $\mathbf{E}_1, \ldots, \mathbf{E}_k$ are continuous vector fields

on the connected open set $U \subset \mathcal{X}$ such that $(\mathbf{E}_1(p), \ldots, \mathbf{E}_k(p)) \in \Omega_p\mathcal{X}$ for all $p \in U$ then $\kappa(p, (\mathbf{E}_1(p), \ldots, \mathbf{E}_k(p)))$ is a constant function of $p \in U$. If $k = 0$ and $\mathcal{X} = \mathcal{P}$ then $\cup_{p\in\mathcal{P}}\Omega_p\mathcal{P}$ is in one-to-one correspondence with $\mathcal{P}$, so an orientation of $\mathcal{P}$ is any mapping $\kappa\colon \mathcal{P} \to \{+1, -1\}$.

- Suppose $\mathcal{X}$ is a k-dimensional manifold, $k \geq 1$, and $\kappa\colon \cup_{p\in\mathcal{X}} \Omega_p\mathcal{X} \to \{+1, -1\}$ is an orientation of $\mathcal{X}$. Suppose U is an open subset of $\mathbb{R}^k$ and $f\colon U \to \mathcal{X}$ is a parameterization of $f(U) \subset \mathcal{X}$. Suppose that for all $q \in U$ and for all $(\mathbf{e}_1, \ldots, \mathbf{e}_k) \in \Omega_{f(q)}\mathcal{X}$ we have that

$$\kappa_{f(q)}(\mathbf{e}_1, \ldots, \mathbf{e}_k) \det[(\mathbf{e}_1, \ldots, \mathbf{e}_k)^T Df(q)] > 0.$$

  Then we say that the parameterization f is *consistent with the orientation* κ.
- Suppose $\mathcal{X}$ is a k-dimensional manifold, $k \geq 1$, and $\partial\mathcal{X}$ is the set of boundary points of $\mathcal{X}$, which is a $(k-1)$-dimensional manifold. If $p \in \partial\mathcal{X}$ let f denote a local parameterization of $\mathcal{X}$ at p. Then define $\mathbf{u}(p) \in T_p\mathcal{X}$ to be the unique unit vector such that $\mathbf{u}(p) \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in T_p\partial\mathcal{X}$ and $\mathbf{u}(p) \cdot Df(\mathbf{0})\hat{\mathbf{e}}_1 > 0$. $\mathbf{u}(p)$ is called the *outward unit normal to* $\partial\mathcal{X}$ *at* p.
- Suppose $\mathcal{X}$ is a k-dimensional manifold, $k \geq 1$, and $\partial\mathcal{X}$ is the set of boundary points of $\mathcal{X}$. Suppose $\kappa\colon \cup_{p\in\mathcal{X}} \Omega_p\mathcal{X} \to \{+1, -1\}$ is an orientation of $\mathcal{X}$. Then the *boundary orientation* on $\partial\mathcal{X}$ it induced by κ is the mapping $\tilde{\kappa}\colon \cup_{p\in\partial X} \Omega_p\partial\mathcal{X} \to \{+1, -1\}$ defined by

$$\tilde{\kappa}_p(\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}) = \kappa_p(\mathbf{u}(p), \mathbf{e}_1, \ldots, \mathbf{e}_{k-1})$$

  for all $p \in \partial\mathcal{X}$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}) \in \Omega_p\partial\mathcal{X}$, where $\mathbf{u}(p)$ is the outward unit normal to $\partial\mathcal{X}$ at p.
- A 0-form on $\mathbb{R}^n$ is an ordinary smooth real-valued function $f\colon U \to \mathbb{R}$, defined on an open subset $U \subset \mathbb{R}^n$.
- If $k \geq 1$ and $f_1, \ldots, f_k$ are 0-forms defined on $U \subset \mathbb{R}^n$ then define the *basic* k*-form* $df_1 \wedge df_2 \wedge \cdots \wedge df_k$ to be the following mapping:

$$(df_1 \wedge df_2 \wedge \cdots \wedge df_k)_x(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det \begin{pmatrix} Df_1(x)\mathbf{v}_1 & \ldots & Df_1(x)\mathbf{v}_k \\ Df_2(x)\mathbf{v}_1 & \ldots & Df_2(x)\mathbf{v}_k \\ \vdots & & \vdots \\ Df_k(x)\mathbf{v}_1 & \ldots & Df_k(x)\mathbf{v}_k \end{pmatrix},$$

  where $x \in U$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$.
- If $k \geq 1$ and $g_1, \ldots, g_m$ are 0-forms defined on $U \subset \mathbb{R}^n$, and $\omega^1, \ldots, \omega^m$ are basic k-forms on U, then $g_1\omega^1 + \cdots + g_m\omega^m$ is called a k*-form on* U. It is a mapping of the arguments $x \in U$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ such that

$$(g_1\omega^1 + \cdots + g_m\omega^m)_x(\mathbf{v}_1, \ldots, \mathbf{v}_k) = g_1(x)\omega^1_x(\mathbf{v}_1, \ldots, \mathbf{v}_k) + \cdots + g_m(x)\omega^m_x(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

- If $\mathcal{P}$ is a 0-dimensional manifold in $\mathbb{R}^n$, $\kappa\colon \mathcal{P} \to \{+1, -1\}$ is an orientation of $\mathcal{P}$, and f is a 0-form defined on the open set $U \subset \mathbb{R}^n$, where $\mathcal{P} \subset U$, then define $\int_{\mathcal{P},\kappa} f = \sum_{p\in\mathcal{P}} f(p)\kappa(p)$.
- Suppose $\mathcal{X}$ is a k-dimensional manifold, $k \geq 1$, and $\kappa\colon \cup_{p\in\mathcal{X}} \Omega_p\mathcal{X} \to \{+1, -1\}$ is an orientation of $\mathcal{X}$. Suppose U is an open subset of $\mathbb{R}^k$ and $\rho\colon U \to \mathcal{X}$ is a parameterization of $\rho(U) \subset \mathcal{X}$ which is consistent with the orientation κ. Suppose $\omega = g_1\omega^1 + \cdots + g_m\omega^m$ is a k-form on an open

subset $V \subset \mathbb{R}^n$ where $\rho(U) \subset V$. Let $u = (u_1, \ldots, u_k)$ be the standard Cartesian coordinates on $\mathbb{R}^k$. Then define

$$\int_{\rho(U),\kappa} \omega = \int_{u \in U} \omega_{\rho(u)}(D\rho(u)\hat{\mathbf{e}}_1, \ldots, D\rho(u)\hat{\mathbf{e}}_k)\, du_1 \ldots du_k.$$

- If f is a 0-form on $\mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ are the standard Cartesian coordinates on $\mathbb{R}^n$ then define df to be the 1-form $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i$. This agrees with our earlier definition of basic 1-forms. In fact, if $f(x) = x_i$ then $df_x(\mathbf{v}) = \mathbf{v}_i$ for all $x \in \mathbb{R}^n$; hence we write $df = dx_i$, omitting any reference to x.
- If $\omega = \sum_{j=1}^m g_j \omega^j$ is a k-form on $U \subset \mathbb{R}^n$, where the basic k-forms ω^j are wedge products of various dx_i's, $i = 1, \ldots, n$, (and hence are independent of x) then define the $(k+1)$-form $d\omega$ by the rule $d\omega = \sum_{j=1}^m dg_j \wedge \omega^j$.

Some Important Facts:

- If ω is a k-form and η is an l-form then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ is a $(k+l)$-form.
- If ω, η, and ζ are differential forms then $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$.
- If ω is a k-form, η and ζ are l-forms, and f and g are 0-forms, then $\omega \wedge (f\eta + g\zeta) = f\omega \wedge \eta + g\omega \wedge \zeta$.
- $d[M(x,y)\, dx + N(x,y)\, dy] = (N_x - M_y)\, dx \wedge\, dy$.
- $d[f(x,y,z)\, dx + g(x,y,z)\, dy + h(x,y,z)\, dz]$
  $= (h_y - g_z)\, dy \wedge\, dz + (f_z - h_x)\, dz \wedge\, dx + (g_x - f_y)\, dx \wedge\, dy$.
  Note that if $F = \langle f, g, h \rangle$ is a vector field then
  $\nabla \times F = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$.
- $d[f(x,y,z)\, dy \wedge\, dz + g(x,y,z)\, dz \wedge\, dx + h(x,y,z)\, dx \wedge\, dy]$
  $= (f_x + g_y + h_z)\, dx \wedge\, dy \wedge\, dz$. Note that if $F = \langle f, g, h \rangle$ is a vector field then $\nabla \cdot F = f_x + g_y + h_z$.
- If $F = \langle f, g, h \rangle$ is a vector field on $\mathbb{R}^3$, $\omega = f(x,y,z)\, dx + g(x,y,z)\, dy + h(x,y,z)\, dz$ is the corresponding 1-form, and $\rho\colon \mathbb{R} \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}$ we have $\omega_{\rho(u)}(D\rho(u)\hat{\mathbf{e}}_1) = F(\rho(u)) \cdot D\rho(u)$.
- If $F = \langle f, g, h \rangle$ is a vector field on $\mathbb{R}^3$, $\omega = f(x,y,z)\, dy \wedge\, dz$ $+\, g(x,y,z)\, dz \wedge\, dx + h(x,y,z)\, dx \wedge\, dy$ is the corresponding 2-form, and $\rho\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}^2$ we have
  $\omega_{\rho(u)}(D\rho(u)\hat{\mathbf{e}}_1, D\rho(u)\hat{\mathbf{e}}_2) = F(\rho(u)) \cdot [D\rho(u)\hat{\mathbf{e}}_1 \times D\rho(u)\hat{\mathbf{e}}_2]$.
- If $\omega = f(x,y,z) dx \wedge dy \wedge dz$ and $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function, then for all $u \in \mathbb{R}^3$ we have $\omega_{\rho(u)}(D\rho(u)\hat{\mathbf{e}}_1, D\rho(u)\hat{\mathbf{e}}_2, D\rho(u)\hat{\mathbf{e}}_3) = f(\rho(u)) \det[D\rho(u)]$.
- **Stokes' Theorem** Suppose $\mathcal{X} \subset \mathbb{R}^n$ is a closed and bounded k-dimensional manifold, $k \geq 1$, with orientation $\kappa\colon \cup_{p \in \mathcal{X}} \Omega_p \mathcal{X} \to \{+1, -1\}$. Suppose $\partial\mathcal{X}$ is the set of boundary points of $\mathcal{X}$, and $\tilde{\kappa}\colon \cup_{p \in \partial X} \Omega_p \partial\mathcal{X} \to \{+1, -1\}$ is the induced boundary orientation. Suppose ω is a $(k-1)$-form on $\mathbb{R}^n$, and $d\omega$ is its exterior derivative (a k-form on $\mathbb{R}^n$). Then

$$\int_{\mathcal{X},\kappa} d\omega = \int_{\partial\mathcal{X},\tilde{\kappa}} \omega.$$

Study Questions:

(1) page 519, number 2, 3, 5.

(2) Suppose $f(x,y) = \frac{3}{2}x^2 + \frac{7}{2}y^2 - 5xy$ and $\mathcal{C}^*$ is the oriented line segment from $(1,3)$ to $(5,2)$. Show that $\int_{\mathcal{C}^*} df = f(5,2) - f(1,3)$ by calculating both sides.

(3) Suppose $\omega = -y\,dx + x\,dy$ is a 1-form on $\mathbb{R}^3$ corresponding to the vector field $F = \langle -y, x, 0\rangle$. Suppose $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0, x^2+y^2+z^2 = 1\}$ is the upper half of a unit sphere. Let $\mathcal{C} = \partial\mathcal{S}$ be the unit circle in the xy plane. Suppose $\mathcal{S}^*$ is oriented such that the "positive side" of the surface is the one away from the origin. Let $\mathcal{C}^*$ have the induced boundary orientation. Show that $\int_{\mathcal{S}^*} d\omega = \int_{\mathcal{C}^*} \omega$ by calculating both sides.

(4) Suppose $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ is a 2-form on $\mathbb{R}^3$. Suppose $\mathcal{R} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$ be the unit sphere together with its interior. Let $\mathcal{S} = \partial\mathcal{R} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ be the unit sphere. Suppose $\mathcal{R}^*$ has the standard orientation of $\mathbb{R}^3$ and $\mathcal{S}^*$ has the induced boundary orientation. Show that $\int_{\mathcal{R}^*} d\omega = \int_{\mathcal{S}^*} \omega$ by calculating both sides.

(5) Suppose $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1\}$ and define the parameterization $\rho(u,v) = \langle 1-u-v, u, v\rangle$, where $\langle u,v\rangle \in \mathbb{R}^2$. Let $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$, and define the orientation $\kappa_p(\mathbf{e}_1, \mathbf{e}_2) = \hat{\mathbf{n}} \cdot \mathbf{e}_1 \times \mathbf{e}_2$, for all $p \in \mathcal{S}$ and all $(\mathbf{e}_1, \mathbf{e}_2) \in \Omega_p\mathcal{S}$. Show that the parameterization ρ is consistent with the orientation κ on $\mathcal{S}$.

(6) Suppose $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1, z \geq 0\}$ and $\mathcal{C} = \{(x,y,0) \in \mathbb{R}^3 \mid x+y = 1\}$. Suppose $\mathcal{S}$ is equipped with the orientation κ defined in problem (5). Show that $\mathbf{u}(p) = \frac{1}{\sqrt{6}}\langle 1,1,-2\rangle$ is the outward unit normal to $\mathcal{C}$. If $\tilde{\kappa}$ is the induced boundary orientation on $\mathcal{C}$, and $\mathbf{e}_1 = \frac{1}{\sqrt{2}}\langle -1,1,0\rangle$ is an orthonormal basis of $T_p\mathcal{C}$, then compute $\tilde{\kappa}_p(\mathbf{e}_1)$.

Solutions to Some Problems:

Problem 3. First we will compute $\int_{\mathcal{C}^*} \omega$, and as always we need a parameterization of $\mathcal{C}^*$ consistent with the orientation. $\mathcal{C}$ is the unit circle in the xy plane, so as discussed in class we parameterize it as follows:

$$\rho(\theta) = \langle \cos\theta, \sin\theta, 0\rangle, \qquad \theta \in [0, 2\pi].$$

From this we calculate

$$D\rho(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle.$$

Recall the ordered basis $(\mathbf{e}_\varrho, \mathbf{e}_\phi, \mathbf{e}_\theta)$ associated to each point in the coordinate domain of spherical coordinates. $(\mathbf{e}_\phi, \mathbf{e}_\theta)$ is a positively oriented orthonormal basis of the tangent space of the unit sphere, $\mathbf{u}(p) = \mathbf{e}_\phi$ is the outward unit normal to $\mathcal{C}$ and $\mathbf{e}_\theta$ is a basis vector of $T_p\mathcal{C}$. Hence ρ, which parameterizes $\mathcal{C}$ via the coordinate θ is consistent with the induced boundary orientation of $\mathcal{C}^*$.

The 1-form $\omega = -y\,dx + x\,dy + 0\,dz$ corresponds to the vector field $F = \langle -y, x, 0\rangle$. By one of our important facts,

$$\begin{aligned}\omega_{\rho(\theta)}(D\rho(\theta)\hat{\mathbf{e}}_1) &= F(\rho(\theta)) \cdot D\rho(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle \\ &= \sin^2\theta + \cos^2\theta = 1.\end{aligned}$$

Now by the definition of the integral of a 1-form over an oriented 1-dimensional manifold we have

$$\int_{\mathcal{C}^*} \omega = \int_0^{2\pi} \omega_{\rho(\theta)}(D\rho(\theta)\hat{\mathbf{e}}_1)\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi.$$

Now we must compute $\int_{\mathcal{S}^*} d\omega$; we begin with a parameterization of $\mathcal{S}^*$. Define $\rho(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ for $(\phi,\theta) \in U = (0,\pi/2)\times(0,2\pi)$. $\rho(U)$ is almost all of $\mathcal{S}$. As we proved in class this parameterization is consistent with the orientation of $\mathcal{S}^*$. Computing we find

$$D\rho(\phi,\theta) = \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{pmatrix}.$$

The 2-form $d\omega = d(-y\,dx + x\,dy) = -dy\wedge dx + dx\wedge dy = 2\,dx\wedge dy = 0\,dy\wedge dz + 0\,dz\wedge dx + 2\,dx\wedge dy$ corresponds to the vector field $F(x,y,z) = \langle 0,0,2\rangle$. By one of our important facts we have

$$\begin{aligned}(d\omega)_{\rho(\phi,\theta)}(D\rho(\phi,\theta)\hat{\mathbf{e}}_1, D\rho(\phi,\theta)\hat{\mathbf{e}}_2) &= F(\rho(\phi,\theta))\cdot D\rho(\phi,\theta)\hat{\mathbf{e}}_1 \times D\rho(\phi,\theta)\hat{\mathbf{e}}_2 \\ &= \det\begin{pmatrix} 0 & 0 & 2 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{pmatrix} \\ &= 2\sin\phi\cos\phi.\end{aligned}$$

Thus by the definition of the integral of a 2-form over an oriented 2-dimensional manifold we have

$$\begin{aligned}\int_{\mathcal{S}^*} d\omega &= \int_U (d\omega)_{\rho(\phi,\theta)}(D\rho(\phi,\theta)\hat{\mathbf{e}}_1, D\rho(\phi,\theta)\hat{\mathbf{e}}_2)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} 2\sin\phi\cos\phi\,d\phi\,d\theta = 2\pi\sin^2\phi\Big|_{\phi=0}^{\phi=\pi/2} = 2\pi.\end{aligned}$$

Thus we found that $\int_{\mathcal{C}^*}\omega = 2\pi$ and $\int_{\mathcal{S}^*} d\omega = 2\pi$, so we have verified that $\int_{\mathcal{S}^*} d\omega = \int_{\mathcal{C}^*}\omega$.

Problem 4. We begin with the calculation of $\int_{\mathcal{S}^*}\omega$. Define the parameterization $\rho(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ for $(\phi,\theta)\in U = (0,\pi)\times(0,2\pi)$. $\rho(U)$ is almost all of $\mathcal{S}$. Recall the ordered basis $(\mathbf{e}_\varrho, \mathbf{e}_\phi, \mathbf{e}_\theta)$ associated to each point in the coordinate domain of spherical coordinates is positively oriented with respect to the standard orientation of $\mathbb{R}^3$. $\mathbf{u}(p) = \mathbf{e}_\varrho$ is the unit outward normal to $\mathcal{S}$. Hence $(\mathbf{e}_\phi, \mathbf{e}_\theta)$ is a positively oriented orthonormal basis of the tangent space of the unit sphere with respect to the induced boundary orientation. As we have seen in class this implies that the parameterization ρ is consistent with the orientation of $\mathcal{S}^*$.

The 2-form $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$ is associated with the vector field $F(x,y,z) = \langle x,y,z\rangle$. By one of our important facts we have

$$\begin{aligned}\omega_{\rho(\phi,\theta)}(D\rho(\phi,\theta)\hat{\mathbf{e}}_1, D\rho(\phi,\theta)\hat{\mathbf{e}}_2) &= F(\rho(\phi,\theta))\cdot D\rho(\phi,\theta)\hat{\mathbf{e}}_1\times D\rho(\phi,\theta)\hat{\mathbf{e}}_2 \\ &= \det\begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0\end{pmatrix} \\ &= \sin\phi.\end{aligned}$$

(Obviously there was a bit of calculation skipped here.) Thus by the definition of the integral of a 2-form over an oriented 2-dimensional manifold we have

$$\begin{aligned}\int_{\mathcal{S}^*} \omega &= \int_U \omega_{\rho(\phi,\theta)}(D\rho(\phi,\theta)\hat{\mathbf{e}}_1, D\rho(\phi,\theta)\hat{\mathbf{e}}_2)\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = -2\pi\cos\phi\Big|_{\phi=0}^{\phi=\pi} = 4\pi.\end{aligned}$$

One of our important facts allows us to compute $d\omega = 3\, dx \wedge\, dy \wedge\, dz$, since $\operatorname{div} F = 3$. Thus

$$\int_{\mathcal{R}^*} d\omega = \int_{\mathcal{R}^*} 3\, dx \wedge\, dy \wedge\, dz = 3(\tfrac{4}{3}\pi) = 4\pi,$$

since the volume of the unit sphere $\mathcal{R}$ is $\frac{4}{3}\pi$. Thus we have computed both integrals and obtained the same answer.

Problem 5. To verify consistency we must check the condition

$$\kappa_{\rho(u,v)}(\mathbf{e}_1, \mathbf{e}_2)\det[(\mathbf{e}_1, \mathbf{e}_2)^T D\rho(u,v)] > 0$$

for all $(\mathbf{e}_1, \mathbf{e}_2) \in \Omega_{\rho(u,v)}\mathcal{S}$. First of all

$$D\rho(u,v) = \begin{pmatrix} \frac{\partial(1-u-v)}{\partial u} & \frac{\partial(1-u-v)}{\partial v} \\ \frac{\partial(u)}{\partial u} & \frac{\partial(u)}{\partial v} \\ \frac{\partial(v)}{\partial u} & \frac{\partial(v)}{\partial v} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose $\mathbf{e}_1 = \langle a_1, b_1, c_1\rangle$ and $\mathbf{e}_2 = \langle a_2, b_2, c_2\rangle$. Then

$$\begin{aligned}\det[(\mathbf{e}_1, \mathbf{e}_2)^T D\rho(u,v)] &= \det\left(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}^T \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} -a_1 + b_1 & -a_1 + c_1 \\ -a_2 + b_2 & -a_2 + c_2 \end{pmatrix} \\ &= (-a_1 + b_1)(-a_2 + c_2) - (-a_2 + b_2)(-a_1 + c_1) \\ &= a_2c_1 - a_1c_2 + a_1b_2 - a_2b_1 + b_1c_2 - b_2c_1.\end{aligned}$$

On the other hand

$$\begin{aligned}\kappa_{\rho(u,v)}(\mathbf{e}_1, \mathbf{e}_2) = \hat{\mathbf{n}} \cdot \mathbf{e}_1 \times \mathbf{e}_2 &= \det\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \\ &= \tfrac{1}{\sqrt{3}}[b_1c_2 - b_2c_1 - (a_1c_2 - a_2c_1) + a_1b_2 - a_2b_1]\end{aligned}$$

Thus

$$\kappa_{\rho(u,v)}(\mathbf{e}_1, \mathbf{e}_2)\det[(\mathbf{e}_1, \mathbf{e}_2)^T D\rho(u,v)] = \tfrac{1}{\sqrt{3}}[b_1c_2 - b_2c_1 + a_2c_1 - a_1c_2 + a_1b_2 - a_2b_1]^2 > 0$$

as we needed to show.

Problem 6. To check that $\mathbf{u}(p)$ is the outward unit normal first note that $\hat{\mathbf{n}} \cdot \mathbf{u}(p) = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle \cdot \frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle = \frac{1}{\sqrt{18}}(1(1) + 1(1) + 1(-2)) = 0$, so $\mathbf{u}(p) \in T_p\mathcal{S}$. Two points on the boundary line $\mathcal{C}$ are $\langle 1, 0, 0\rangle$ and $\langle 0, 1, 0\rangle$, so a tangent vector to this line is $\langle 0,1,0\rangle - \langle 1,0,0\rangle = \langle -1,1,0\rangle$. Also $\mathbf{u}(p)\cdot\langle -1,1,0\rangle = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle\cdot\langle -1,1,0\rangle = 0$, so $\mathbf{u}(p)$ is perpendicular to $T_p\mathcal{C}$. Also since $\mathcal{S}$ has $z \geq 0$ whereas the z-component of $\mathbf{u}(p)$ is negative, we have that $\mathbf{u}(p)$ points "outward" from $\mathcal{S}$.

To compute $\tilde{\kappa}_p(\mathbf{e}_1)$ we use the definition of the induced boundary orientation:

$$\begin{aligned}\tilde{\kappa}_p(\mathbf{e}_1) &= \kappa_p(\mathbf{u}(p),\mathbf{e}_1) = \hat{\mathbf{n}}\cdot\mathbf{u}(p)\times\mathbf{e}_1 = \hat{\mathbf{n}}\times\mathbf{u}(p)\cdot\mathbf{e}_1\\ &= \tfrac{1}{\sqrt{3}}\langle 1,1,1\rangle\times\tfrac{1}{\sqrt{6}}\langle 1,1,-2\rangle\cdot\mathbf{e}_1 = \tfrac{1}{\sqrt{18}}\langle -3,3,0\rangle\cdot\mathbf{e}_1\\ &= \tfrac{1}{\sqrt{2}}\langle -1,1,0\rangle\cdot\mathbf{e}_1\end{aligned}$$

This dot product is either $+1$ or -1 according as $\mathbf{e}_1 = \frac{\pm 1}{\sqrt{2}}\langle -1,1,0\rangle$. So the positive direction on $\mathcal{C}$ is in the direction of $\frac{1}{\sqrt{2}}\langle -1,1,0\rangle$.