MATH 550, VECTOR ANALYSIS, EXTRA HOMEWORK 3 AND SOLUTIONS

Problem 0. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Show that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{b}(\mathbf{a} \cdot \mathbf{c})$.

Solution: Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle, \mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Then $\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$, and

$$
\begin{aligned}
&(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \\
&= \begin{pmatrix} (a_3 b_1 - a_1 b_3) c_3 - (a_1 b_2 - a_2 b_1) c_2 \\ (a_1 b_2 - a_2 b_1) c_1 - (a_2 b_3 - a_3 b_2) c_3 \\ (a_2 b_3 - a_3 b_2) c_2 - (a_3 b_1 - a_1 b_3) c_1 \end{pmatrix} = \begin{pmatrix} -a_1(b_2 c_2 + b_3 c_3) + b_1(a_2 c_2 + a_3 c_3) \\ -a_2(b_1 c_1 + b_3 c_3) + b_2(a_1 c_1 + a_3 c_3) \\ -a_3(b_1 c_1 + b_2 c_2) + b_3(a_1 c_1 + a_2 c_2) \end{pmatrix} \\
&= \begin{pmatrix} -a_1(b_1 c_1 + b_2 c_2 + b_3 c_3) + b_1(a_1 c_1 + a_2 c_2 + a_3 c_3) \\ -a_2(b_1 c_1 + b_2 c_2 + b_3 c_3) + b_2(a_1 c_1 + a_2 c_2 + a_3 c_3) \\ -a_3(b_1 c_1 + b_2 c_2 + b_3 c_3) + b_3(a_1 c_1 + a_2 c_2 + a_3 c_3) \end{pmatrix} \\
&= -\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1 c_1 + b_2 c_2 + b_3 c_3) + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} (a_1 c_1 + a_2 c_2 + a_3 c_3) \\
&= -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{b}(\mathbf{a} \cdot \mathbf{c}).
\end{aligned}
$$

Suppose $\hat{\mathbf{u}}$ is a unit vector in the abstract vector space V, and $\theta \in \mathbb{R}$. For every vector $\mathbf{r}$ in V define:

$$R(\theta, \hat{\mathbf{u}})\mathbf{r} = \mathbf{r} \cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})(1 - \cos\theta) + \hat{\mathbf{u}} \times \mathbf{r} \sin\theta.$$

Suppose $\mathbf{v}, \mathbf{w} \in V$.

Problem 1. Show that $[R(\theta, \hat{\mathbf{u}})\mathbf{v}] \cdot [R(\theta, \hat{\mathbf{u}})\mathbf{w}] = \mathbf{v} \cdot \mathbf{w}$.

Solution: We use direct calculation:

$$
\begin{aligned}
&R(\theta, \hat{\mathbf{u}})\mathbf{v} \cdot R(\theta, \hat{\mathbf{u}})\mathbf{w} \\
&= [\mathbf{v} \cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v})(1 - \cos\theta) + \hat{\mathbf{u}} \times \mathbf{v} \sin\theta] \cdot [\mathbf{w} \cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{w})(1 - \cos\theta) + \hat{\mathbf{u}} \times \mathbf{w} \sin\theta] \\
&= \mathbf{v} \cdot \mathbf{w} \cos^2\theta + (\hat{\mathbf{u}} \cdot \mathbf{v})(\hat{\mathbf{u}} \cdot \mathbf{w}) \cos\theta(1 - \cos\theta) + \mathbf{v} \cdot \hat{\mathbf{u}} \times \mathbf{w} \cos\theta \sin\theta \\
&\quad + (\hat{\mathbf{u}} \cdot \mathbf{w})(\hat{\mathbf{u}} \cdot \mathbf{v}) \cos\theta(1 - \cos\theta) + (\hat{\mathbf{u}} \cdot \mathbf{v})(\hat{\mathbf{u}} \cdot \mathbf{w})(1 - \cos\theta)^2 \\
&\quad + \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \times \mathbf{w}(\hat{\mathbf{u}} \cdot \mathbf{v})(1 - \cos\theta) \sin\theta + \hat{\mathbf{u}} \times \mathbf{v} \cdot \mathbf{w} \cos\theta \sin\theta \\
&\quad + \hat{\mathbf{u}} \times \mathbf{v} \cdot \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{w})(1 - \cos\theta) \sin\theta + (\hat{\mathbf{u}} \times \mathbf{v}) \cdot (\hat{\mathbf{u}} \times \mathbf{w}) \sin^2\theta
\end{aligned}
$$

Because $\hat{\mathbf{u}}$ is perpendicular to $\hat{\mathbf{u}} \times \mathbf{v}$ and $\hat{\mathbf{u}} \times \mathbf{w}$, we have $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \times \mathbf{w} = \mathbf{0}$ and $\hat{\mathbf{u}} \times \mathbf{v} \cdot \hat{\mathbf{u}} = \mathbf{0}$. Also by the triple product identity $\mathbf{v} \cdot \hat{\mathbf{u}} \times \mathbf{w} = \hat{\mathbf{u}} \times \mathbf{w} \cdot \mathbf{v} = \hat{\mathbf{u}} \cdot \mathbf{w} \times \mathbf{v} = -\hat{\mathbf{u}} \cdot \mathbf{v} \times \mathbf{w}$, and $\hat{\mathbf{u}} \times \mathbf{v} \cdot \mathbf{w} = \hat{\mathbf{u}} \cdot \mathbf{v} \times \mathbf{w}$. Hence the terms involving $\sin\theta\cos\theta$ cancel out. Collecting terms a bit we get:

$$R(\theta, \hat{\mathbf{u}})\mathbf{v} \cdot R(\theta, \hat{\mathbf{u}})\mathbf{w} = \mathbf{v} \cdot \mathbf{w} \cos^2\theta + (\hat{\mathbf{u}} \cdot \mathbf{v})(\hat{\mathbf{u}} \cdot \mathbf{w}) \sin^2\theta + (\hat{\mathbf{u}} \times \mathbf{v}) \cdot (\hat{\mathbf{u}} \times \mathbf{w}) \sin^2\theta$$

Using the triple product identity and problem 0 we get
$(\hat{\mathbf{u}}\times\mathbf{v})\cdot(\hat{\mathbf{u}}\times\mathbf{w}) = [(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}]\cdot\mathbf{w} = [-\hat{\mathbf{u}}(\mathbf{v}\cdot\hat{\mathbf{u}})+\mathbf{v}(\hat{\mathbf{u}}\cdot\hat{\mathbf{u}})]\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w}-(\hat{\mathbf{u}}\cdot\mathbf{v})(\hat{\mathbf{u}}\cdot\mathbf{w})$.
Substituting this into the previous expression we get

$$R(\theta,\hat{\mathbf{u}})\mathbf{v}\cdot R(\theta,\hat{\mathbf{u}})\mathbf{w} = \mathbf{v}\cdot\mathbf{w}(\cos^2\theta+\sin^2\theta) = \mathbf{v}\cdot\mathbf{w}.$$

Using problem 0 we can show the following *Jacobi identity* holds:
$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = \mathbf{0}$. To see this we compute:

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= -\mathbf{a}(\mathbf{b}\cdot\mathbf{c})+\mathbf{b}(\mathbf{a}\cdot\mathbf{c})\\
(\mathbf{c}\times\mathbf{a})\times\mathbf{b} &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{a}(\mathbf{c}\cdot\mathbf{b})\\
(\mathbf{b}\times\mathbf{c})\times\mathbf{a} &= -\mathbf{b}(\mathbf{c}\cdot\mathbf{a})+\mathbf{c}(\mathbf{b}\cdot\mathbf{a}).
\end{aligned}$$

Adding both sides we clearly get a zero vector on the right-hand-side.

Problem 2. Show that $R(\theta,\hat{\mathbf{u}})(\mathbf{v}\times\mathbf{w}) = [R(\theta,\hat{\mathbf{u}})\mathbf{v}]\times[R(\theta,\hat{\mathbf{u}})\mathbf{w}]$.
Solution: As before we directly calculate:

$$\begin{aligned}
&R(\theta,\hat{\mathbf{u}})\mathbf{v}\times R(\theta,\hat{\mathbf{u}})\mathbf{w}\\
&= [\mathbf{v}\cos\theta+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})(1-\cos\theta)+\hat{\mathbf{u}}\times\mathbf{v}\sin\theta]\times[\mathbf{w}\cos\theta+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{w})(1-\cos\theta)+\hat{\mathbf{u}}\times\mathbf{w}\sin\theta]\\
&= \mathbf{v}\times\mathbf{w}\cos^2\theta+(\mathbf{v}\times\hat{\mathbf{u}})(\hat{\mathbf{u}}\cdot\mathbf{w})\cos\theta(1-\cos\theta)+\mathbf{v}\times(\hat{\mathbf{u}}\times\mathbf{w})\cos\theta\sin\theta\\
&\quad+(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})\cos\theta(1-\cos\theta)+\hat{\mathbf{u}}\times\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})(\hat{\mathbf{u}}\cdot\mathbf{w})(1-\cos\theta)^2\\
&\quad+\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})(1-\cos\theta)\sin\theta+(\hat{\mathbf{u}}\times\mathbf{v})\times\mathbf{w}\cos\theta\sin\theta\\
&\quad+(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{w})(1-\cos\theta)\sin\theta+(\hat{\mathbf{u}}\times\mathbf{v})\times(\hat{\mathbf{u}}\times\mathbf{w})\sin^2\theta\\
&= [\mathbf{v}\times\mathbf{w}-(\mathbf{v}\times\hat{\mathbf{u}})(\hat{\mathbf{u}}\cdot\mathbf{w})-(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})]\cos^2\theta+(\hat{\mathbf{u}}\times\mathbf{v})\times(\hat{\mathbf{u}}\times\mathbf{w})\sin^2\theta\\
&\quad+[\mathbf{v}\times(\hat{\mathbf{u}}\times\mathbf{w})-\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})+(\hat{\mathbf{u}}\times\mathbf{v})\times\mathbf{w}-(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{w})]\cos\theta\sin\theta\\
&\quad+[(\mathbf{v}\times\hat{\mathbf{u}})(\hat{\mathbf{u}}\cdot\mathbf{w})+(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})]\cos\theta\\
&\quad+[\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v})+(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{w})]\sin\theta
\end{aligned}$$

We must work on each of these terms. The first term is (using problem 0):

$$\begin{aligned}
\mathbf{v}\times\mathbf{w}-(\mathbf{v}\times\hat{\mathbf{u}})(\hat{\mathbf{u}}\cdot\mathbf{w})-(\hat{\mathbf{u}}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\mathbf{v}) &= \mathbf{v}\times\mathbf{w}+[-\mathbf{v}(\mathbf{w}\cdot\hat{\mathbf{u}})+\mathbf{w}(\mathbf{v}\cdot\hat{\mathbf{u}})]\times\hat{\mathbf{u}}\\
&= \mathbf{v}\times\mathbf{w}+[(\mathbf{v}\times\mathbf{w})\times\hat{\mathbf{u}}]\times\hat{\mathbf{u}}\\
&= \mathbf{v}\times\mathbf{w}-(\mathbf{v}\times\mathbf{w})(\hat{\mathbf{u}}\cdot\hat{\mathbf{u}})+\hat{\mathbf{u}}[(\mathbf{v}\times\mathbf{w})\cdot\hat{\mathbf{u}}]\\
&= \hat{\mathbf{u}}[\hat{\mathbf{u}}\cdot(\mathbf{v}\times\mathbf{w})].
\end{aligned}$$

The second term is (using problem 0 and the triple product identity):

$$\begin{aligned}
(\hat{\mathbf{u}}\times\mathbf{v})\times(\hat{\mathbf{u}}\times\mathbf{w}) &= -\hat{\mathbf{u}}[\mathbf{v}\cdot(\hat{\mathbf{u}}\times\mathbf{w})]+\mathbf{v}[\hat{\mathbf{u}}\cdot(\hat{\mathbf{u}}\times\mathbf{w})] = \hat{\mathbf{u}}[\mathbf{v}\cdot(\mathbf{w}\times\hat{\mathbf{u}})] = \hat{\mathbf{u}}[(\mathbf{v}\times\mathbf{w})\cdot\hat{\mathbf{u}}]\\
&= \hat{\mathbf{u}}[\hat{\mathbf{u}}\cdot(\mathbf{v}\times\mathbf{w})].
\end{aligned}$$

The third term is (using problem 0 and the Jacobi identity):

$$
\begin{aligned}
&\mathbf{v} \times (\hat{\mathbf{u}} \times \mathbf{w}) + (\hat{\mathbf{u}} \times \mathbf{v}) \times \mathbf{w} - \hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{w})(\hat{\mathbf{u}} \cdot \mathbf{v}) - (\hat{\mathbf{u}} \times \mathbf{v}) \times \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{w}) \\
&= (\mathbf{w} \times \hat{\mathbf{u}}) \times \mathbf{v} + (\hat{\mathbf{u}} \times \mathbf{v}) \times \mathbf{w} + \{\hat{\mathbf{u}} \times [-\mathbf{v}(\hat{\mathbf{u}} \cdot \mathbf{w}) + \mathbf{w}(\hat{\mathbf{u}} \cdot \mathbf{v})]\} \times \hat{\mathbf{u}} \\
&= (\mathbf{w} \times \hat{\mathbf{u}}) \times \mathbf{v} + (\hat{\mathbf{u}} \times \mathbf{v}) \times \mathbf{w} + \{\hat{\mathbf{u}} \times [(\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}}]\} \times \hat{\mathbf{u}} \\
&= (\mathbf{w} \times \hat{\mathbf{u}}) \times \mathbf{v} + (\hat{\mathbf{u}} \times \mathbf{v}) \times \mathbf{w} - \hat{\mathbf{u}}\{[(\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}}] \cdot \hat{\mathbf{u}}\} + [(\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}}](\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) \\
&= (\mathbf{w} \times \hat{\mathbf{u}}) \times \mathbf{v} + (\hat{\mathbf{u}} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}} \\
&= \mathbf{0}.
\end{aligned}
$$

The fourth term is:

$$
\begin{aligned}
(\mathbf{v} \times \hat{\mathbf{u}})(\hat{\mathbf{u}} \cdot \mathbf{w}) + (\hat{\mathbf{u}} \times \mathbf{w})(\hat{\mathbf{u}} \cdot \mathbf{v}) &= \hat{\mathbf{u}} \times [-\mathbf{v}(\hat{\mathbf{u}} \cdot \mathbf{w}) + \mathbf{w}(\hat{\mathbf{u}} \cdot \mathbf{v})] = \hat{\mathbf{u}} \times [(\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}}] \\
&= -[(\mathbf{v} \times \mathbf{w}) \times \hat{\mathbf{u}}] \times \hat{\mathbf{u}} \\
&= (\mathbf{v} \times \mathbf{w})(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) - \hat{\mathbf{u}}[(\mathbf{v} \times \mathbf{w}) \cdot \hat{\mathbf{u}}] \\
&= \mathbf{v} \times \mathbf{w} - \hat{\mathbf{u}}[\hat{\mathbf{u}} \cdot (\mathbf{v} \times \mathbf{w})].
\end{aligned}
$$

The fifth term is:

$$
\begin{aligned}
&\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{w})(\hat{\mathbf{u}} \cdot \mathbf{v}) + (\hat{\mathbf{u}} \times \mathbf{v}) \times \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{w}) \\
&\qquad = [\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{w}) - \mathbf{w}(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})](\hat{\mathbf{u}} \cdot \mathbf{v}) + [-\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v}) + \mathbf{v}(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})](\hat{\mathbf{u}} \cdot \mathbf{w}) \\
&\qquad = -\mathbf{w}(\hat{\mathbf{u}} \cdot \mathbf{v}) + \mathbf{v}(\hat{\mathbf{u}} \cdot \mathbf{w}) \\
&\qquad = (\mathbf{w} \times \mathbf{v}) \times \hat{\mathbf{u}} \\
&\qquad = \hat{\mathbf{u}} \times (\mathbf{v} \times \mathbf{w}).
\end{aligned}
$$

Plugging all these expressions in we get:

$$
\begin{aligned}
&R(\theta, \hat{\mathbf{u}})\mathbf{v} \times R(\theta, \hat{\mathbf{u}})\mathbf{w} \\
&\qquad = \hat{\mathbf{u}}[\hat{\mathbf{u}} \cdot (\mathbf{v} \times \mathbf{w})] + \{\mathbf{v} \times \mathbf{w} - \hat{\mathbf{u}}[\hat{\mathbf{u}} \cdot (\mathbf{v} \times \mathbf{w})]\} \cos\theta + \hat{\mathbf{u}} \times (\mathbf{v} \times \mathbf{w}) \sin\theta \\
&\qquad = R(\theta, \hat{\mathbf{u}})(\mathbf{v} \times \mathbf{w}).
\end{aligned}
$$