EXTRA HOMEWORK 2 WITH SOLUTIONS

Suppose $\hat{\mathbf{u}}$ is a unit vector in the abstract vector space V, and $\theta \in \mathbb{R}$. For every vector $\mathbf{r}$ in V define:

$$R(\theta, \hat{\mathbf{u}})\mathbf{r} = \mathbf{r}\cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})(1 - \cos\theta) + \hat{\mathbf{u}} \times \mathbf{r}\sin\theta.$$

Answer the following:

Problem 1 Show that $R(-\theta, -\hat{\mathbf{u}})\mathbf{r} = R(\theta, \hat{\mathbf{u}})\mathbf{r}$. Draw a picture illustrating this fact.

Solution: Note that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, so

$$\begin{aligned} R(-\theta, -\hat{\mathbf{u}})\mathbf{r} &= \mathbf{r}\cos(-\theta) + (-\hat{\mathbf{u}})[(-\hat{\mathbf{u}}) \cdot \mathbf{r}][1 - \cos(-\theta)] + (-\hat{\mathbf{u}}) \times \mathbf{r}\sin(-\theta) \\ &= \mathbf{r}\cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})(1 - \cos\theta) + \hat{\mathbf{u}} \times \mathbf{r}\sin\theta \\ &= R(\theta, \hat{\mathbf{u}})\mathbf{r}. \end{aligned}$$

For the picture see Figure 1a).

Problem 2 If $\theta = \pi$ show that $R(\pi, -\hat{\mathbf{u}})\mathbf{r} = R(\pi, \hat{\mathbf{u}})\mathbf{r}$. Draw a picture illustrating this fact.

Solution: Computing we have:

$$\begin{aligned} R(\pi, -\hat{\mathbf{u}})\mathbf{r} &= \mathbf{r}\cos(\pi) + (-\hat{\mathbf{u}})[(-\hat{\mathbf{u}}) \cdot \mathbf{r}][1 - \cos(\pi)] + (-\hat{\mathbf{u}}) \times \mathbf{r}\sin(\pi) \\ &= -\mathbf{r} + 2\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r}) \\ &= \mathbf{r}\cos(\pi) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})[1 - \cos(\pi)] + \hat{\mathbf{u}} \times \mathbf{r}\sin(\pi) \\ &= R(\pi, \hat{\mathbf{u}})\mathbf{r}. \end{aligned}$$

For the picture see Figure 1b).

Problem 3 If $\pi \leq \theta < 2\pi$ show that $R(\theta, \hat{\mathbf{u}})\mathbf{r} = R(2\pi - \theta, -\hat{\mathbf{u}})\mathbf{r}$. Draw a picture illustrating this fact.

Solution: Because of the periodicity of the sine and cosine we have

$$\begin{aligned} &R(2\pi - \theta, -\hat{\mathbf{u}})\mathbf{r} \\ &\quad = \mathbf{r}\cos(2\pi - \theta) + (-\hat{\mathbf{u}})[(-\hat{\mathbf{u}}) \cdot \mathbf{r}][1 - \cos(2\pi - \theta)] + (-\hat{\mathbf{u}}) \times \mathbf{r}\sin(2\pi - \theta) \\ &\quad = \mathbf{r}\cos(-\theta) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})[1 - \cos(-\theta)] - \hat{\mathbf{u}} \times \mathbf{r}\sin(-\theta) \\ &\quad = \mathbf{r}\cos(\theta) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{r})[1 - \cos(\theta)] + \hat{\mathbf{u}} \times \mathbf{r}\sin(\theta) \\ &\quad = R(\theta, \hat{\mathbf{u}})\mathbf{r}. \end{aligned}$$

For the picture see Figure 1c).

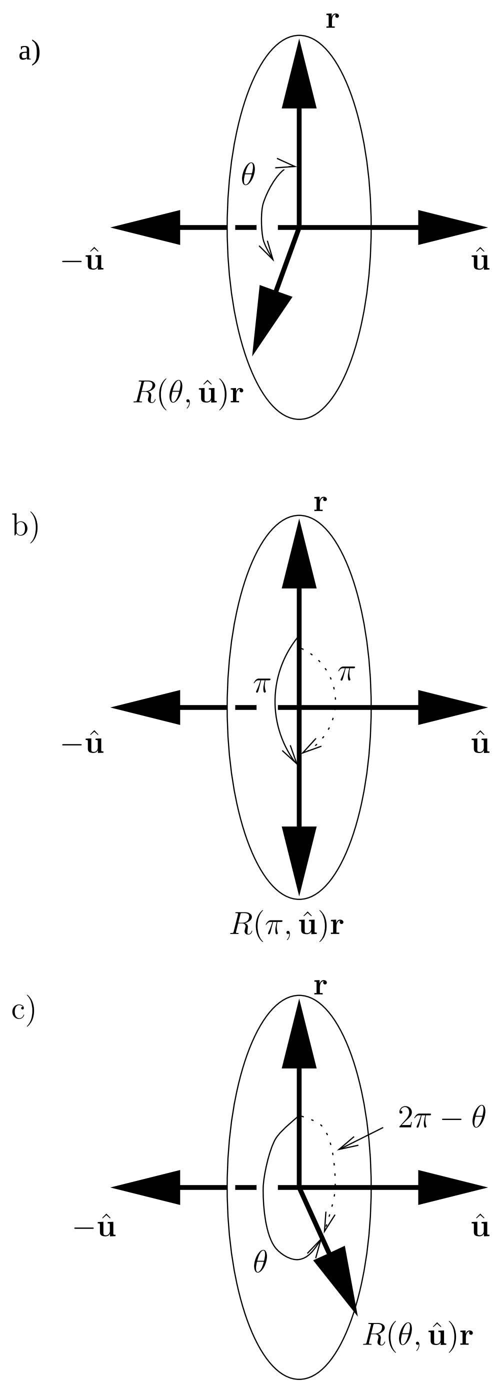
a)
$\mathbf{r}$
θ
$-\hat{\mathbf{u}}$
$\hat{\mathbf{u}}$
$R(\theta, \hat{\mathbf{u}})\mathbf{r}$
b)
$\mathbf{r}$
π
π
$-\hat{\mathbf{u}}$
$\hat{\mathbf{u}}$
$R(\pi, \hat{\mathbf{u}})\mathbf{r}$
c)
$\mathbf{r}$
$2\pi - \theta$
$-\hat{\mathbf{u}}$
$\hat{\mathbf{u}}$
θ
$R(\theta, \hat{\mathbf{u}})\mathbf{r}$

Figure 1