MATH 550, VECTOR ANALYSIS, EXTRA HOMEWORK 1 AND SOLUTIONS

Suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of the abstract vector space V, and define

$$A = \begin{pmatrix} -\frac{\sqrt{6}}{4} & -\frac{2\sqrt{2}+\sqrt{6}}{8} & \frac{-\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{4} & \frac{2\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{3}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

Let $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)A$ be another orthonormal basis. Define $\mathbf{v} = \mathbf{e}'_1 - 2\mathbf{e}'_2 + 3\mathbf{e}'_3$ and $\mathbf{w} = -2\mathbf{e}'_1 - \mathbf{e}'_2 - 4\mathbf{e}'_3$. Answer the following:

Problem 1. Find the components of $\mathbf{v}$ and $\mathbf{w}$ relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Solution: We know that $\mathbf{v} = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)\langle 1, -2, 3\rangle$ and $\mathbf{w} = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)\langle -2, -1, -4\rangle$. (Recall that $\langle a, b, c\rangle$ is an abbreviation for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, which takes up too much room.) Using the equation $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)A$ we have $\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)A\langle 1, -2, 3\rangle$ and $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)A\langle -2, -1, -4\rangle$. Computing we get

$$A\langle 1, -2, 3\rangle = \begin{pmatrix} -\frac{\sqrt{6}}{4} & -\frac{2\sqrt{2}+\sqrt{6}}{8} & \frac{-\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{4} & \frac{2\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{3}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}+6\sqrt{6}}{8} \\ -\frac{7\sqrt{2}+6\sqrt{6}}{8} \\ \frac{-8+3\sqrt{3}}{4} \end{pmatrix},$$

$$A\langle -2, -1, -4\rangle = \begin{pmatrix} -\frac{\sqrt{6}}{4} & -\frac{2\sqrt{2}+\sqrt{6}}{8} & \frac{-\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{4} & \frac{2\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{3}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{6\sqrt{2}-3\sqrt{6}}{8} \\ \frac{2\sqrt{2}+13\sqrt{6}}{8} \\ \frac{1-4\sqrt{3}}{4} \end{pmatrix}.$$

Thus $\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\langle \frac{\sqrt{2}+6\sqrt{6}}{8}, -\frac{7\sqrt{2}+6\sqrt{6}}{8}, \frac{-8+3\sqrt{3}}{4}\rangle$ and
$\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\langle \frac{6\sqrt{2}-3\sqrt{6}}{8}, \frac{2\sqrt{2}+13\sqrt{6}}{8}, \frac{1-4\sqrt{3}}{4}\rangle$.

Problem 2. Compute $\mathbf{v} \cdot \mathbf{w}$ using the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and also compute $\mathbf{v} \cdot \mathbf{w}$ using the basis $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$. Do you get the same answer?

Solution: Using the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ we have $\mathbf{v} \cdot \mathbf{w} = (\frac{\sqrt{2}+6\sqrt{6}}{8})(\frac{6\sqrt{2}-3\sqrt{6}}{8}) + (-\frac{7\sqrt{2}+6\sqrt{6}}{8})(\frac{2\sqrt{2}+13\sqrt{6}}{8}) + (\frac{-8+3\sqrt{3}}{4})(\frac{1-4\sqrt{3}}{4}) = -12$. Using the basis $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ we have $\mathbf{v} \cdot \mathbf{w} = (1)(-2) + (-2)(-1) + (3)(-4) = -12$. Clearly we get the same answer.

Problem 3. Compute $\mathbf{v} \times \mathbf{w}$ using the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and also compute $\mathbf{v} \times' \mathbf{w}$ using the basis $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$. Do you get the same answer?

Solution: Using the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ we have $\mathbf{v} \times \mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\langle(-\frac{7\sqrt{2}+6\sqrt{6}}{8})(\frac{1-4\sqrt{3}}{4}) - (\frac{-8+3\sqrt{3}}{4})(\frac{2\sqrt{2}+13\sqrt{6}}{8}), (\frac{-8+3\sqrt{3}}{4})(\frac{6\sqrt{2}-3\sqrt{6}}{8}) - (\frac{\sqrt{2}+6\sqrt{6}}{8})(\frac{1-4\sqrt{3}}{4}), (\frac{\sqrt{2}+6\sqrt{6}}{8})(\frac{2\sqrt{2}+13\sqrt{6}}{8}) - (-\frac{7\sqrt{2}+6\sqrt{6}}{8})(\frac{6\sqrt{2}-3\sqrt{6}}{8})\rangle = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\langle \frac{-9\sqrt{2}+30\sqrt{6}}{8}, \frac{-\sqrt{2}+10\sqrt{6}}{8}, \frac{28+5\sqrt{3}}{4}\rangle$.

Using the basis $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ we have $\mathbf{v} \times' \mathbf{w} = (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')\langle(-2)(-4) - (3)(-1), (3)(-2) - (1)(-4), (1)(-1) - (-2)(-2)\rangle = (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')\langle 11, -2, -5\rangle$. To see if $\mathbf{v} \times' \mathbf{w}$ is the same as $\mathbf{v} \times \mathbf{w}$ we need to express both of them in the same basis. It is easiest to convert $\mathbf{v} \times' \mathbf{w}$ into the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$: $\mathbf{v} \times' \mathbf{w} = (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')\langle 11, -2, -5\rangle = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)A\langle 11, -2, -5\rangle$. Computing we have

$$A\langle 11, -2, -5\rangle = \begin{pmatrix} -\frac{\sqrt{6}}{4} & -\frac{2\sqrt{2}+\sqrt{6}}{8} & \frac{-\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{4} & \frac{2\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{3}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 11 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{9\sqrt{2}-30\sqrt{6}}{8} \\ \frac{\sqrt{2}-10\sqrt{6}}{8} \\ \frac{-28-5\sqrt{3}}{4} \end{pmatrix}.$$

Thus $\mathbf{v} \times' \mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\langle \frac{9\sqrt{2}-30\sqrt{6}}{8}, \frac{\sqrt{2}-10\sqrt{6}}{8}, \frac{-28-5\sqrt{3}}{4}\rangle$, and hence $\mathbf{v} \times' \mathbf{w} = -\mathbf{v} \times \mathbf{w}$.

Problem 4. Compute $\det(A)$.

Solution:

$$\begin{aligned} \det(A) &= (-\tfrac{\sqrt{6}}{4}) \det \begin{pmatrix} \frac{2\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} - (-\tfrac{2\sqrt{2}+\sqrt{6}}{8}) \det \begin{pmatrix} -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}+2\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} \end{pmatrix} \\ &\quad + (\tfrac{-\sqrt{2}+2\sqrt{6}}{8}) \det \begin{pmatrix} -\frac{\sqrt{6}}{4} & \frac{2\sqrt{2}-\sqrt{6}}{8} \\ -\frac{1}{2} & \frac{3}{4} \end{pmatrix} \\ &= -1. \end{aligned}$$

Thus the bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ determine opposite orientations. This explains why in problem 3 we did not get the same answer. Note that the orientations of the bases did not matter in problem 2.