MATH 550, PROBLEM 18 FROM PAGE 429

Problem 18. Find the surface area of the helicoid parameterized by $\mathbf{X}\colon D \to \mathbb{R}^3$, $D = (0,1) \times (0, 2\pi n)$, n a positive integer, where $\mathbf{X}(r,\theta) = \langle r\cos\theta, r\sin\theta, \theta\rangle$, $(r,\theta) \in D$.

Solution:

$$D\mathbf{X}(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 0 & 1 \end{pmatrix}.$$

According to the formula (6) on page 425 the surface area should be computed by the rule

$$\int_0^{2\pi n}\int_0^1 \|D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_1 \times D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_2\|\,dr\,d\theta.$$

Computing the cross product we get

$$D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_1 \times D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_2 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ r \end{pmatrix}.$$

Hence $\|D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_1 \times D\mathbf{X}(r,\theta)\hat{\mathbf{e}}_2\| = \sqrt{1+r^2}$. Thus the surface area of the helicoid is

$$\int_0^{2\pi n}\int_0^1 \sqrt{1+r^2}\,dr\,d\theta = 2\pi n\int_0^1 \sqrt{1+r^2}\,dr.$$

This last integral is done by a trigonometric substitution followed by integration by parts, or by look up in the table of integrals. Thus

$$\int \sqrt{1+r^2}\,dr = \frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln|r + \sqrt{1+r^2}| + C.$$

Therefore

$$\int_0^1 \sqrt{1+r^2}\,dr = \frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2},$$

and hence the area is $\pi n[\sqrt{2} + \ln(1+\sqrt{2})]$.