PRACTICE HOMEWORK SET #5 FOR MATH 550 (EUCLIDEAN SPACE, COORDINATE SYSTEMS)

Problem #1: Assume the earth is a sphere of radius 6378 km. Cape Canaveral (the typical launching point of space shuttles) is at latitude 28.47° north and longitude 80.54° west. Assume that the Cartesian coordinates of a point ρ km from the center of the earth and at longitude θ (east is positive, west is negative) and latitude ϕ (north is positive, south is negative) is $(x, y, z) = (\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, \rho\sin\phi)$. Here we are giving these coordinates in terms of a system rotating with the earth, with origin at the center of the earth. The x axis points toward the equator (0° latitude) at 0° longitude. The y axis points toward the equator (0° latitude) at 90° (east) longitude. The z axis points toward the north pole. Let $\mathcal{T}_{(\rho,\theta,\phi)}$ denote the set of three dimensional arrows with base point at (ρ, θ, ϕ) and terminal point some point in space. Consider the following basis of $\mathcal{T}_{(\rho,\theta,\phi)}$:

$$\mathbf{v}_1 = \frac{\partial}{\partial\rho}\begin{pmatrix}\rho\cos\theta\cos\phi\\ \rho\sin\theta\cos\phi\\ \rho\sin\phi\end{pmatrix}, \mathbf{v}_2 = \frac{\partial}{\partial\theta}\begin{pmatrix}\rho\cos\theta\cos\phi\\ \rho\sin\theta\cos\phi\\ \rho\sin\phi\end{pmatrix}, \mathbf{v}_3 = \frac{\partial}{\partial\phi}\begin{pmatrix}\rho\cos\theta\cos\phi\\ \rho\sin\theta\cos\phi\\ \rho\sin\phi\end{pmatrix}.$$

So $(\rho, \theta, \phi) = (6378, -80.54°, 28.47°)$ is Cape Canaveral and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ (evaluated at $(\rho, \theta, \phi) = (6378, -80.54°, 28.47°)$) is a basis of $\mathcal{T}_{(6378,-80.54°,28.47°)}$. This basis is not orthonormal. Nevertheless we can think of this basis as defining a coordinate system with perpendicular coordinate axes whose origin P_{cc} is at Cape Canaveral. The point in space which is $P_{cc} \boldsymbol{+} (\mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 + \mathbf{v}_3 x_3)$ has coordinates (x_1, x_2, x_3) in this coordinate system. Here we are using $\boldsymbol{+}$ as an operation we discussed in class, 'point in space'$\boldsymbol{+}$'vector'=another 'point in space'. Suppose the International Space Station (ISS) is 400 km above the earth's surface, and at some particular time has longitude $\theta = -70°$ and latitude $\phi = -20°$ (over Iquique, Chile). Find the coordinates of the ISS in this coordinate system at Cape Canaveral. Should the ISS be visible on the horizon at Cape Canaveral?

Problem #2: (#1 continued.) Ideally the ISS should be oriented in space so that its solar panels can get direct sunlight (except when it is in the earth's shadow). This means that astronauts on the ISS cannot always see the earth whenever they look out of a particular window. Suppose at the same instant of time when the ISS has coordinates

$(\rho, \theta, \phi) = (6778, -70°, -20°)$ there is a basis $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ fixed to the ISS given by

$$\mathbf{w}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix},$$

in terms of the x, y, z axes fixed at the center of the earth. This is an orthonormal basis. The ISS coordinate system has its origin P_{iss} at the position of the ISS and coordinate axes in the directions of the basis vectors $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Suppose a space shuttle is launched from Cape Canaveral and at this instant has coordinates (x_1, x_2, x_3) relative to the Cape Canaveral coordinate system. Find the coordinates of the space shuttle relative to the ISS coordinate system at that instant of time. This information would be useful to the ISS astronauts who would like to know which window to look out of to get a visual on the space shuttle.

Problem #3: (#2 continued.) Let P_{ss} denote the point in space where the space shuttle is. Let $\mathcal{T}_{(6778,-70°,-20°)}$ be the arrows based at P_{iss}. Suppose $\mathbf{v} \in \mathcal{T}_{(6378,-80.54°,28.47°)}$ such that $P_{ss} = P_{cc} + \mathbf{v}$, and suppose $\mathbf{w} \in \mathcal{T}_{(6778,-70°,-20°)}$ such that $P_{ss} = P_{iss} + \mathbf{w}$. In problem #2 we were given $\mathbf{v}$ and were asked to find $\mathbf{w}$. In this problem, suppose one is given $\mathbf{w} = \mathbf{w}_1 y_1 + \mathbf{w}_2 y_2 + \mathbf{w}_3 y_3$. Find $\mathbf{v} = \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 + \mathbf{v}_3 x_3$. (Be careful, since $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is not orthonormal, so $x_i = \mathbf{v}_i^*(\mathbf{v})$, where $(\mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*)$ is the dual basis to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.)