**Problem 1 (Facts about SU(2)).** Let SU(2) denote the set of all $2 \times 2$ matrices $U$ with complex entries such that $U^\dagger U = UU^\dagger = I$ and det $U = 1$. Show that for all real $\theta$ and real unit vectors $u$ the matrix $H(e^{i\theta}, u)$ is in SU(2). Show that any matrix $U = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in SU(2) satisfies $a_{21} = -\bar{a}_{12}, a_{22} = \bar{a}_{11}$, and $|a_{11}|^2 + |a_{12}|^2 = 1$, and hence $U = H(e^{i\theta}, u)$ for some pair $(e^{i\theta}, u)$ as above. (Hint: Use the relation $a_{11}a_{22} - a_{12}a_{21} = 1$ and the first row of the equation $U^\dagger U = I$.) Show that if $U_1, U_2 \in SU(2)$ then $U_1^{-1}U_2 \in SU(2)$.

**Problem 2 (Axis and angle for products of rotations).** Show how to find a real unit vector $u$ and a real angle $0 \leq \theta \leq \pi$ such that $H(e^{i\theta/2}, u) = H(e^{i\theta_1/2}, u_1)H(e^{i\theta_2/2}, u_2)$.

**Problem 3 (Active and passive rotations).** Let $R(e^{i\theta}, u) = uu^T + [I - uu^T] \cos \theta + [u \times \sin \theta$, where

$$[u \times \sin \theta] = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

provided $u = \langle u_1, u_2, u_3 \rangle$. Show that $R(e^{i\theta}, u) \in SO(3)$. Suppose $B \in SO(3)$. Show that $R(e^{i\theta}, Bu)B = BR(e^{i\theta}, u)$. If $B = (b_1, b_2, b_3)$ is interpreted as a right-handed orthonormal basis of $\mathbb{R}^3$ then $R(e^{i\theta}, Bu)$ is considered an active rotation whereas $R(e^{i\theta}, u)$ is considered a passive rotation. The effect of these two types of rotations on the basis $B$ is the same (that is what you are asked to show above). If $B' = R(e^{i\theta}, Bu)(b_1', b_2', b_3') = (b_1', b_2', b_3')$ is the rotated basis, and $v = \langle v_1, v_2, v_3 \rangle$, then show that $R(e^{i\eta}, B'v)R(e^{i\theta}, Bu)B = BR(e^{i\eta}, u)v$. Thus active and passive rotations are composed in opposite orders.

**Problem 4 (Rigid Motions).** If $A \in SO(3)$ and $b, x \in \mathbb{R}^3$ then let $M(b, A)$ and $\tilde{x}$ denote

$$M(b, A) = \begin{pmatrix} 1 & b^T \\ b & A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{21} & a_{22} \\ b_3 & a_{31} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

$M(b, A)$ is called a rigid motion of $\mathbb{R}^3$. Show that $M(b, A)\tilde{x} = (b + Ax)$. This is interpreted as the effect of the rigid motion on the point $x$ in space. Show that $M(b, A)^{-1} = M(-ATb, AT)$, and that $M(b_1, A_1)M(b_2, A_2) = M(b_1 + A_1b_2, A_1A_2)$.

**Problem 5 (T-matrices as rigid motions).** Show that $T_0(l) = M(\hat{e}_3, l, R(e^{i\pi}, \hat{e}_1))$, $T_1(\cos \theta) = M(0, R(e^{i\theta}, \hat{e}_2)R(e^{i\pi}, \hat{e}_3))$, and $T_2(e^{i\phi}) = M(0, R(e^{i\phi}, \hat{e}_3))$, where $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ denote the three columns of the identity matrix.