MATH 728B, TRANSLATIONAL AND ROTATIONAL INVARIANCE II

Recall we defined the operator $\mathcal{S}_k(j)$ to be the matrix $\frac{\hbar}{2}\boldsymbol{\sigma}_k$ acting in the jth spin variable. In the tensor notation, this definition is expressed as

$$\mathcal{S}_k(j)s = \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_N=0}^{1} s(\sigma_1, \ldots, \sigma_N)\boldsymbol{\alpha}_{\sigma_1} \otimes \cdots \otimes \frac{\hbar}{2}\boldsymbol{\sigma}_k \boldsymbol{\alpha}_{\sigma_j} \otimes \cdots \otimes \boldsymbol{\alpha}_{\sigma_N},$$

i.e. $\mathcal{S}_k(j) = I \otimes \cdots \otimes \frac{\hbar}{2}\boldsymbol{\sigma}_k \otimes \cdots \otimes I$, where $\frac{\hbar}{2}\boldsymbol{\sigma}_k$ replaces the jth copy of I. Recall that $\mathcal{S}_k = \sum_{j=1}^{N} \mathcal{S}_k(j)$, $k = 1, 2, 3$. Now we are prepared to state the following.

Lemma. *Suppose the spin function s*

$$s = \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_N=0}^{1} s(\sigma_1, \ldots, \sigma_N)\boldsymbol{\alpha}_{\sigma_1} \otimes \cdots \otimes \boldsymbol{\alpha}_{\sigma_N},$$

is an eigenfunction of $\mathcal{S}^2$ with eigenvalue $\hbar^2 S(S+1)$ (where $S \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{N}{2}\}$) and is also an eigenfunction of $\mathcal{S}_3$ with eigenvalue $\hbar(S - \sigma)$ (where $0 \le \sigma \le 2S$). Define $\mathfrak{h} = H(e^{i\theta/2}, \mathbf{u}) \otimes \cdots \otimes H(e^{i\theta/2}, \mathbf{u})$ (N times), where $\theta \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^3$ is a unit vector. Define $\tilde{s}$ by the rule: $\tilde{s} = \mathfrak{h}s$, i.e.

$$\tilde{s} = \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_N=0}^{1} s(\sigma_1, \ldots, \sigma_N) H(e^{i\theta/2}, \mathbf{u})\boldsymbol{\alpha}_{\sigma_1} \otimes \cdots \otimes H(e^{i\theta/2}, \mathbf{u})\boldsymbol{\alpha}_{\sigma_N},$$

If $\hat{\mathbf{e}}_3 = \langle 0, 0, 1\rangle$ and $\mathbf{y} = R(e^{i\theta}, \mathbf{u})\hat{\mathbf{e}}_3 = \langle y_1, y_2, y_3\rangle$ then define the rotated multi-electron spin operator by $\frac{\hbar}{2}\mathcal{X}(\mathbf{y}) = \mathcal{S}_1 y_1 + \mathcal{S}_2 y_2 + \mathcal{S}_3 y_3$. Then $\tilde{s}$ is an eigenfunction of $\mathcal{S}^2$ with the same eigenvalue as s, and $\tilde{s}$ is an eigenfunction of $\frac{\hbar}{2}\mathcal{X}(\mathbf{y})$ with eigenvalue $\hbar(S - \sigma)$.

Proof. Note that

$$\begin{aligned}
\frac{\hbar}{2}\mathcal{X}(\mathbf{y})\mathfrak{h} &= \sum_{k=1}^{3} \mathcal{S}_k \mathfrak{h} y_k = \sum_{k=1}^{3}\sum_{j=1}^{N} (I \otimes \cdots \otimes \frac{\hbar}{2}\boldsymbol{\sigma}_k \otimes \cdots \otimes I)\mathfrak{h} y_k \\
&= \sum_{j=1}^{N} (I \otimes \cdots \otimes \sum_{k=1}^{3} \frac{\hbar}{2}\boldsymbol{\sigma}_k y_k \otimes \cdots \otimes I)\mathfrak{h} \\
&= \sum_{j=1}^{N} H(e^{i\theta/2}, \mathbf{u}) \otimes \cdots \otimes \frac{\hbar}{2}X(\mathbf{y})H(e^{i\theta/2}, \mathbf{u}) \otimes \cdots \otimes H(e^{i\theta/2}, \mathbf{u}) \\
&= \sum_{j=1}^{N} H(e^{i\theta/2}, \mathbf{u}) \otimes \cdots \otimes H(e^{i\theta/2}, \mathbf{u})\frac{\hbar}{2}X(\hat{\mathbf{e}}_3) \otimes \cdots \otimes H(e^{i\theta/2}, \mathbf{u}) \\
&= \mathfrak{h}\sum_{j=1}^{N} (I \otimes \cdots \otimes \frac{\hbar}{2}\boldsymbol{\sigma}_3 \otimes \cdots \otimes I) \\
&= \mathfrak{h}\mathcal{S}_3.
\end{aligned}$$

Thus $\frac{\hbar}{2}\mathcal{X}(\mathbf{y})\tilde{s} = \frac{\hbar}{2}\mathcal{X}(\mathbf{y})\mathfrak{h}s = \mathfrak{h}\mathcal{S}_3 s = \mathfrak{h}\hbar(S-\sigma)s = \hbar(S-\sigma)\tilde{s}$, as claimed.

Suppose $R(e^{i\theta},\mathbf{u})\hat{\mathbf{e}}_k = \sum_{m=1}^{3} R_k^m \hat{\mathbf{e}}_m$. Then one shows that $\mathcal{S}^2\mathfrak{h} = \mathfrak{h}\mathcal{S}^2$ as follows.

$$
\begin{aligned}
\mathfrak{h}\mathcal{S}^2 &= \mathfrak{h}\sum_{k=1}^{3}\left(\sum_{j=1}^{N}\mathcal{S}_k(j)\right)\left(\sum_{l=1}^{N}\mathcal{S}_k(l)\right) \\
&= \sum_{k=1}^{3}\left[\sum_{j=1}^{N}\mathfrak{h}\left(I\overset{1}{\otimes}\dots\overset{j-1}{\otimes}\frac{\hbar}{2}\boldsymbol{\sigma}_k\overset{j}{\otimes}\dots\overset{N-1}{\otimes}I\right)\mathfrak{h}^{\dagger}\right]\left[\sum_{l=1}^{N}\mathfrak{h}\left(I\overset{1}{\otimes}\dots\overset{l-1}{\otimes}\frac{\hbar}{2}\boldsymbol{\sigma}_k\overset{l}{\otimes}\dots\overset{N-1}{\otimes}I\right)\right] \\
&= \sum_{k=1}^{3}\left[\sum_{j=1}^{N}\left(I\overset{1}{\otimes}\dots\overset{j-1}{\otimes}\frac{\hbar}{2}X(R(e^{i\theta},\mathbf{u})\hat{\mathbf{e}}_k)\overset{j}{\otimes}\dots\overset{N-1}{\otimes}I\right)\right]\cdot \\
&\qquad\cdot\left[\sum_{l=1}^{N}\left(I\overset{1}{\otimes}\dots\overset{l-1}{\otimes}\frac{\hbar}{2}X(R(e^{i\theta},\mathbf{u})\hat{\mathbf{e}}_k)\overset{l}{\otimes}\dots\overset{N-1}{\otimes}I\right)\right]\mathfrak{h} \\
&= \sum_{k=1}^{3}\left[\sum_{j=1}^{N}\sum_{m=1}^{3}R_k^m\left(I\overset{1}{\otimes}\dots\overset{j-1}{\otimes}\frac{\hbar}{2}\boldsymbol{\sigma}_m\overset{j}{\otimes}\dots\overset{N-1}{\otimes}I\right)\right]\cdot \\
&\qquad\cdot\left[\sum_{l=1}^{N}\sum_{n=1}^{3}R_k^n\left(I\overset{1}{\otimes}\dots\overset{l-1}{\otimes}\frac{\hbar}{2}\boldsymbol{\sigma}_n\overset{l}{\otimes}\dots\overset{N-1}{\otimes}I\right)\right]\mathfrak{h} \\
&= \sum_{k=1}^{3}\left[\sum_{m=1}^{3}R_k^m\mathcal{S}_m\right]\left[\sum_{n=1}^{3}R_k^n\mathcal{S}_n\right]\mathfrak{h} \\
&= \sum_{m=1}^{3}\sum_{n=1}^{3}\left[\sum_{k=1}^{3}R_k^m R_k^n\right]\mathcal{S}_m\mathcal{S}_n\mathfrak{h} \\
&= \mathcal{S}^2\mathfrak{h}
\end{aligned}
$$

The claim follows from this as before. $\square$