A VECTOR FIELD THAT IS NOT A GRADIENT

From our last class. We worked on an example

\[ \mathbf{G}(x, y) = \left\langle \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right\rangle, \]

defined for \((x, y) \in \mathcal{D} = \{(x, y) \mid (x, y) \neq 0\}\). We sought a \(C^1\)-function \(g: \mathcal{D} \to \mathbb{R}\) such that \(\mathbf{G} = \nabla g\) on \(\mathcal{D}\). After working very hard integrating both sides of

\[ g_x = \frac{-y}{\sqrt{x^2 + y^2}} \]

equation we obtained

\[ g(x, y) = -y \ln(x + \sqrt{x^2 + y^2}) + C(y). \]

Since \(g\) is \(C^1\) then \(C'(y)\) should be continuous in \(y\). From this point the next step is to insert this result into the second equation \(g_y = \frac{x}{\sqrt{x^2 + y^2}}\) to see what it tells us about \(C(y)\). This process leads to the doubtful looking equality

\[ C'(y) = \frac{x}{\sqrt{x^2 + y^2}} + \ln(x + \sqrt{x^2 + y^2}) + \frac{y^2}{(x + \sqrt{x^2 + y^2})(x^2 + y^2)^{3/2}}. \]

Could it be true that the right-hand-side is independent of \(x\) despite its appearance (i.e. some surprising identity comes to the rescue)? If so then it shouldn’t hurt to take the limit of both sides as \(x \to \infty\) (holding \(y \neq 0\) fixed in the process); but this leads to \(C'(y) = \infty\), for each \(y \neq 0\). This contradiction shows that there is no \(C^1\) function \(g\) so that \(\mathbf{G} = \nabla g\) on \(\mathcal{D}\).

Actually there is a quick way to see this without all the hard calculations. Note that \(\mathbf{G}(x, y)\) is a unit vector for all \((x, y) \in \mathcal{D}\) and \(\mathbf{G}(ax, ay) = \mathbf{G}(x, y)\) for all \((x, y) \in \mathcal{D}\) and \(a > 0\). For a fixed pair \((x_0, y_0) \in \mathcal{D}\) the ray \(\{(ax_0, ay_0) \mid a > 0\}\) starts at the origin and proceeds in a straight line passing through the point \((x_0, y_0)\) and continues on to infinity. \(\mathbf{G}(x, y)\) is constant on any such ray, and is perpendicular to it, i.e. \((x_0, y_0) \cdot \mathbf{G}(x_0, y_0) = 0\). If it were true that \(\mathbf{G} = \nabla g\) on \(\mathcal{D}\) for some \(C^1\)-function \(g: \mathcal{D} \to \mathbb{R}\) then each such ray would be a level set of \(g\). Now consider two such rays, say both near the positive \(x\)-axis. These two rays get closer together near the origin and so the gradient vector \(\mathbf{G}(x, y) = \nabla g(x, y)\) should get longer and longer as \((x, y)\) gets nearer the origin. (Remember gradient vectors are longer where level sets are closely spaced.) But this doesn’t happen, since \(\mathbf{G}(x, y)\) is always a unit vector. This contradiction shows that \(g\) cannot exist.
A general method of showing \( G \) is not a gradient. The geometric argument we just gave cannot work in other situations; it depends on particular properties of this example, such as \( G(ax, ay) = G(x, y) \) for all \((x, y) \in D\) and \(a > 0\). However there is a generally applicable argument. Let \( G(x, y) = (X(x, y), Y(x, y)) \) and suppose there is a \( C^1 \) function \( g: D \to \mathbb{R} \) such that \( X = g_x \) and \( Y = g_y \) on \( D \). If \( X, Y \) are \( C^1 \) functions (this happens a lot, such as in our example above) then \( X_y = g_{xy} \) and \( Y_x = g_{yx} \) are continuous. By one of our theorems (under these conditions) we must have \( X_y = Y_x \) on \( D \). The equality \( X_y = Y_x \) is something pretty easy to check. In our example:

\[
\frac{\partial}{\partial y} \left( \frac{-y}{\sqrt{x^2 + y^2}} \right) - \frac{\partial}{\partial x} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{-1}{\sqrt{x^2 + y^2}} + \frac{(-y)(-\frac{1}{2})2y}{(x^2 + y^2)^{3/2}} + \frac{-1}{\sqrt{x^2 + y^2}} + \frac{(-x)(-\frac{1}{2})2x}{(x^2 + y^2)^{3/2}} = \frac{-1}{\sqrt{x^2 + y^2}}.
\]

Clearly this is not 0, so (again) such a function \( g \) cannot exist.

Another example. We can dream up a vector field like \( G \) but fixing the failure \( G \) had to get longer near the origin.

\[
F(x, y) = \frac{1}{\sqrt{x^2 + y^2}} G(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right),
\]

defined for \((x, y) \in D\). We seek a \( C^1 \) function \( f: D \to \mathbb{R} \) such that \( F = \nabla f \) on \( D \). If we use the quick check to see if \( X_y - Y_x = 0 \) we find

\[
\frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) - \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{-2}{x^2 + y^2} = 0.
\]

Does this mean \( f \) does exist? Maybe, maybe not. The quick check method is good for showing \( f \) does not exist when \( X_y - Y_x \neq 0 \), but when \( X_y - Y_x = 0 \) it doesn’t prove that \( f \) exists as a \( C^1 \) function on the entire domain \( D \). Also it doesn’t help us at all to find \( f \). For that we need our original procedure.
Following our usual procedure we integrate both sides of \( f(x, y) = \frac{-y}{x^2 + y^2} \) with respect to \( x \). First we assume \( y \neq 0 \), and integrate:

\[
f(x, y) = \int \frac{-y}{x^2 + y^2} \, dx = \int \frac{-dx}{y(\frac{x}{y})^2 + 1} = -\int \frac{du}{u^2 + 1}
\]

\[
= -\tan^{-1}(u) + \text{(constant relative to } x) = -\tan^{-1}(\frac{x}{y}) + C_1(y).
\]

Now insert this into the equation \( f_y = \frac{x}{x^2 + y^2} \) to get

\[
\frac{\partial}{\partial y} \left[ -\tan^{-1}(\frac{x}{y}) + C_1(y) \right] = \frac{-1}{\left(\frac{x}{y}\right)^2 + 1} \cdot \frac{(-x)}{y^2} + C_1'(y)
\]

\[
= \frac{x}{x^2 + y^2} + C_1'(y) = \frac{x^2}{x^2 + y^2}.
\]

This implies \( C_1'(y) = 0 \) for all \( y \neq 0 \), i.e. \( C_1(y) \) is constant on \((-\infty, 0)\) and on \((0, \infty)\). So \( C_1(y) = C_1(1) \) for all \( y > 0 \) and \( C_1(y) = C_1(-1) \) for all \( y < 0 \). Thus we have that \( f(x, y) = -\tan^{-1}(\frac{x}{y}) + C_1(\text{sign}(y)) \). This doesn’t make sense when \( y = 0 \).

So what does this mean? Have we found \( f \)? Not everywhere on \( \mathcal{D} \), since we made the assumption that \( y \neq 0 \). So now relax that assumption on \( y \) and instead assume \( x \neq 0 \), and start over integrating both sides of the second equation \( f_y = \frac{x}{x^2 + y^2} \) with respect to \( y \) in a similar way as before.

\[
f(x, y) = \int \frac{x}{x^2 + y^2} \, dy = \int \frac{x \, dy}{x^2[1 + (\frac{y}{x})^2]} = \int \frac{du}{1 + u^2}
\]

\[
= \tan^{-1}(u) + \text{(constant with respect to } y)
\]

\[
= \tan^{-1}(\frac{y}{x}) + C_2(x).
\]

Now insert this into the first equation \( f_x = \frac{-y}{x^2 + y^2} \) to get

\[
\frac{\partial}{\partial x} \left[ \tan^{-1}(\frac{y}{x}) + C_2(x) \right] = \frac{1}{(\frac{y}{x})^2 + 1} \cdot \frac{(-y)}{x^2} + C_2'(x)
\]

\[
= \frac{-y}{y^2 + x^2} + C_2'(x) = \frac{-y}{x^2 + y^2}.
\]

This implies \( C_2'(x) = 0 \) for all \( x \neq 0 \), i.e. \( C_2(x) = C_2(\text{sign}(x)) \). So we have that \( f(x, y) = \tan^{-1}(\frac{y}{x}) + C_2(\text{sign}(x)) \) for all \( x \neq 0 \). Again this still doesn’t work everywhere in \( \mathcal{D} \).

But we *have* found a patchwork of formulas for \( f(x, y) \) capable of being defined at any point of \( \mathcal{D} \), provided the four constants \( C_1(1), C_1(-1), C_2(1), \text{ and } C_2(-1) \) can be chosen appropriately so that in the four quadrants the formulas match. For example, if \( C_2(1) = 0 \) then
\( f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \) for all \( x > 0 \). This is one of the formulas for the \( \theta \) angle in polar coordinates. We have the following result.

**Fact.** Assume \( x > 0 \) and \( y \neq 0 \). Then

\[
\tan^{-1}\left(\frac{y}{x}\right) = \begin{cases} 
-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} & \text{if } y > 0, \\
-\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2} & \text{if } y < 0.
\end{cases}
\]

So the choice \( C_2(1) = 0 \) and matching in quadrants I and IV leads to the choices \( C_1(\pm 1) = \pm \frac{\pi}{2} \). The proof of this fact is based on the relations

\[
-\frac{\pi}{2} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{2}, \quad \text{for all } y
\]

\[
0 < -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} < \frac{\pi}{2}, \quad \text{for } y > 0, \text{ and}
\]

\[
-\frac{\pi}{2} < -\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2} < 0, \quad \text{for } y < 0,
\]

and the fact that on the interval \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\) the function \( \sin \theta \) is one-to-one. So it is enough to show that

\[
\sin[\tan^{-1}\left(\frac{y}{x}\right)] = \begin{cases} 
\sin[-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}] & \text{if } y > 0, \\
\sin[-\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2}] & \text{if } y < 0.
\end{cases}
\]

We know that \( \sin[\tan^{-1}\left(\frac{y}{x}\right)] = \frac{y}{\sqrt{x^2+y^2}} \) and

\[
\cos[-\tan^{-1}\left(\frac{x}{y}\right)] = \cos[\tan^{-1}\left(\frac{x}{y}\right)] = \cos[\tan^{-1}\left(\frac{x}{|y|}\right)] = \frac{|y|}{\sqrt{x^2+y^2}}.
\]

Using the addition formula \( \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \) we have

\[
\sin[-\tan^{-1}\left(\frac{x}{y}\right) \pm \frac{\pi}{2}] = \sin[-\tan^{-1}\left(\frac{x}{y}\right)] \cos \frac{\pi}{2} \pm \cos[-\tan^{-1}\left(\frac{x}{y}\right)] \sin \frac{\pi}{2}
\]

\[
= \pm \frac{|y|}{\sqrt{x^2+y^2}}.
\]

We use the upper sign when \( y > 0 \) and the lower sign when \( y < 0 \), so \( \pm |y| = y \). This finishes the proof.

So with these choices we have defined the following:

\[
f(x, y) = \begin{cases} 
\tan^{-1}\left(\frac{y}{x}\right), & x > 0, \\
-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}, & y > 0, \\
-\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2}, & y < 0.
\end{cases}
\]

But this still doesn’t define \( f(x, 0) \) for \( x < 0 \). For fixed \( x < 0 \) we compute the limits of \( f(x, y) \) as \( y \to 0^+ \) and as \( y \to 0^- \) as follows.

\[
\lim_{{y \to 0^+}} f(x, y) = \lim_{{y \to 0^+}} [-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}] = -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi;
\]

\[
\lim_{{y \to 0^-}} f(x, y) = \lim_{{y \to 0^-}} [-\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2}] = -(-\frac{\pi}{2}) - \frac{\pi}{2} = -\pi.
\]
and
\[ \lim_{y \to 0^{-}} f(x, y) = \lim_{y \to 0^{-}} \left[ -\tan^{-1}\left( \frac{x}{y} \right) - \frac{\pi}{2} \right] = -(\frac{\pi}{2}) - \frac{\pi}{2} = -\pi. \]

Because these two limits are different, we cannot make a choice of \( f(x, 0) \) that would make \( f \) continuous on all \( D \). We would have come to the same conclusion if we had chosen \( C_2(1) \) to be something other than \( 0 \) (just add \( C_2(1) \) to the above \( f \)). Thus there is no \( C^1 \) function \( f \) defined on all \( D \) so that \( \mathbf{F} = \nabla f \) on all \( D \). In this case we can however find an \( f \) that works on large portions of \( D \). The problem is with \( D \) more than it is with the formula defining \( \mathbf{F} \) on \( D \).

The function \( f \) we found everywhere on \( D \) except on the negative \( x \)-axis is in fact the \( \theta \) angle we defined in polar coordinates. This is because of the following result.

**Fact.** If \( y \neq 0 \) then
\[ -\tan^{-1}\left( \frac{x}{y} \right) + \text{sign}(y) \frac{\pi}{2} = \text{sign}(y) \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}. \]

The proof is another exercise in trigonometric functions and their inverses, which you might want to try to generate yourself.