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PARTIAL UNCONDITIONALITY

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Abstract. J. Elton proved that for $\delta \in (0,1]$ there exists $K(\delta) < \infty$ such that every normalized weakly null sequence in a Banach space admits a subsequence (x_i) with the following property: if $a_i \in [-1,1]$ for all $i \in \mathbb{N}$ and $E \subset \{i \in \mathbb{N} : |a_i| \geq \delta\}$, then $\|\sum_{i \in E} a_i x_i\| \leq K(\delta)\|\sum_i a_i x_i\|$. It is unknown if $\sup_{\delta>0} K(\delta) < \infty$. This problem turns out to be closely related to the question whether every infinite-dimensional Banach space contains a quasi-greedy basic sequence. The notion of a quasi-greedy basic sequence was introduced recently by S. V. Konyagin and V. N. Temlyakov. We present an extension of Elton's result which includes Schreier unconditionality. The proof involves a basic framework which we show can be also employed to prove other partial unconditionality results including that of convex unconditionality due to Argyros, Mercourakis and Tsarpalias. Various constants of partial unconditionality are defined and we investigate the relationships between them. We also explore the combinatorial problem underlying the $\sup_{\delta>0} K(\delta) < \infty$ problem and show that $\sup_{\delta>0} K(\delta) \geq 5/4$.

Dedicated to Haskell Rosenthal on the occasion of his 65^{th} birthday

Contents

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1. Introduction

Given a weakly null, normalized sequence in a Banach space, can we pass to a subsequence that is a basic sequence and is in some sense close to being unconditional? There are various ways in which one can make this vague question precise, and in many situations one has a positive answer. There are important cases, however, for which the corresponding question is still open. In this paper we will study such questions and provide some partial answers. We will also revisit known results and discuss the relationship (e.g. duality) between the various notions of partial unconditionality.

As usual, we denote by c_{00} the space of scalar sequences that are eventually zero. Given a basic sequence (x_i) in a Banach space and $\delta \in (0,1]$, we say (x_i) is *δ-near-unconditional with constant C* if its basis constant is at most C and

$$\Big\| \sum_{i\in E} a_i x_i \Big\| \leq C \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\| \tag{1}$$

for all $(a_i)\in \mathrm{c}_{00}$ with $|a_i|\leq 1$ for all $i\in\mathbb{N}$, and for all $E\subset\{i\in\mathbb{N} : |a_i|\geq\delta\}$. Roughly speaking, this says that we are allowed to project vectors onto sets of co-ordinates with "large" coefficients. A basic sequence is called *δ-near-unconditional* if for some C it is δ-near-unconditional with constant C; it is called *near-unconditional* if it is δ-near-unconditional for all $\delta\in(0,1]$. The following result is due to J. Elton.

Theorem 1.1 (Elton [9])**.** *For each $\delta \in (0,1]$, every normalized, weakly null sequence has a δ-near-unconditional subsequence. In particular, every normalized, weakly null sequence has a near-unconditional subsequence.*

For each $\delta\in(0,1]$ let $K(\delta)$ be the infimum of the set of real numbers K such that every normalized, weakly null sequence has a δ-near-unconditional subsequence with constant K. An upper bound of order $\log\big(1/\delta\big)$ for $K(\delta)$ follows from the proof of Theorem 1.1 presented in [20]. This was first pointed out by Dilworth, Kalton and Kutzarova [10]. It is unknown whether there is in fact a uniform upper bound.

Problem 1.2. *Let K be the function defined above. Is* $\sup_{\delta>0} K(\delta)<\infty$?

Additional motivation for this problem comes from approximation theory. A positive answer to Problem 1.2 would imply the existence of a quasi-greedy basic sequence in every infinite-dimensional Banach space. A basic sequence (x_i) in a Banach space is called *quasi-greedy* if there exists a constant C such that for all $\delta > 0$ and for all $(a_i) \in \mathrm{c}_{00}$, (1) above holds with $E = \{i \in \mathbb{N} : |a_i| \geq \delta\}$. In other words, we can project with a *uniform* constant onto sets consisting of *all* co-ordinates with "large" coefficients. This concept was introduced by Konyagin and Temlyakov [16]. One of the main results in this paper, Theorem 2.1, gives a positive answer to Problem 1.2 under some additional assumptions on the sets of co-ordinates onto which we can project.

We will now place the above notions in a wider context. We will explain the term 'partial unconditionality' and discuss further examples. Let (x_i) be a sequence of non-zero vectors in a Banach space. Then (x_i) is a basic sequence with constant C if and only if (1) holds for all $(a_i) \in \mathrm{c}_{00}$ and whenever $E = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. Moreover, (x_i) is an *unconditional basic sequence* if and only if (1) holds for all $(a_i) \in \mathrm{c}_{00}$ and for all finite subsets E of $\mathbb{N}$. Thus for a basic sequence we can uniformly project onto initial segments of $\mathbb{N}$, whereas for an unconditional sequence we can uniformly project onto *all* finite (or indeed infinite) subsets of $\mathbb{N}$. By partial unconditionality we mean a property of a sequence of non-zero vectors in a Banach space that lies between these two extremes. We next describe one way in which this idea can be formalized.

Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$. Given a sequence (x_i) of non-zero vectors in a Banach space, we say that (x_i) is *$\mathcal{F}$-unconditional with constant C* if (1) holds for all $(a_i) \in \mathrm{c}_{00}$ and for all finite sets E such that either $E \in \mathcal{F}$ or E is an initial segment of $\mathbb{N}$. Our opening question can now be made precise: Does every normalized, weakly null sequence have an $\mathcal{F}$-unconditional subsequence?

If $\mathcal{F} = \emptyset$, then (x_i) is $\mathcal{F}$-unconditional with constant C if and only if it is a basic sequence with constant C. It is well known that for any $\epsilon > 0$ every normalized, weakly null sequence has a subsequence that is a basic sequence with constant $1+\epsilon$. On the other hand if $\mathcal{F}$ is the set of all finite subsets of $\mathbb{N}$, then (x_i) is $\mathcal{F}$-unconditional with constant C if and only if it is an unconditional sequence with constant C. In this case our question has a negative answer: in 1974 Maurey and Rosenthal constructed a Banach space with a normalized, weakly null basis which has no unconditional subsequence. Note that by Rosenthal's ℓ_1-theorem [24], if a space contains no normalized, weakly null sequence, then it contains ℓ_1 and, in particular, an unconditional basic sequence. Thus, given a collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$, a more general question would be to ask if every infinite-dimensional Banach space contains an $\mathcal{F}$-unconditional sequence. For unconditional sequences it was not until 1993 that the more general question was also answered in the negative by Gowers and Maurey [14]. They constructed a Banach space that contains no unconditional basic sequence.

Because of the Maurey-Rosenthal and Gowers-Maurey counterexamples it is an interesting problem to search for non-trivial examples of partial unconditionality that lead to positive answers to the questions we raised above. As it happens such examples occur naturally in various contexts. We give two examples which are relevant in the study of spreading models and asymptotic structures in Banach space theory. A finite subset E of $\mathbb{N}$ is a *Schreier set* if $|E|\leq \min E$. The collection of all Schreier sets is denoted by $\mathcal{S}_1$. A sequence of non-zero vectors in a Banach space is called *Schreier-unconditional* if it is $\mathcal{S}_1$-unconditional. The following result was announced in [18], a proof is given in [21].

Theorem 1.3. *For each $\epsilon>0$, every normalized weakly null sequence in a Banach space has a Schreier-unconditional subsequence with constant $2+\epsilon$.*

One could generalize Schreier-unconditionality by considering higher-order Schreier families that were introduced by Alspach and Odell [2] and by Alspach and Argyros [1]. For example $\mathcal{S}_2$ can be defined as the collection of disjoint unions $\bigcup_{i=1}^n F_i$ of Schreier sets $F_1,\dots,F_n$ with $\{\min F_1,\dots,\min F_n\}\in\mathcal{S}_1$. Unfortunately, the questions corresponding to $\mathcal{S}_2$ already have negative answers: the basis in the example of Maurey and Rosenthal has no $\mathcal{S}_2$-unconditional subsequence, and the space of Gowers and Maurey contains no $\mathcal{S}_2$-unconditional basic sequence. However, it is worth mentioning two positive results here. Let α be a countable ordinal and let $\mathcal{S}_\alpha$ denote the Schreier family of order α. It is shown in [3] that if the normalized weakly null sequence (x_i) is an ℓ_1^α-spreading model, then (x_i) admits an $\mathcal{S}_\alpha$-unconditional subsequence. Moreover, in [12] it is shown that an $\mathcal{S}_\alpha$-unconditional normalized weakly null sequence in $C(\mathcal{S}_\alpha)$ admits an unconditional subsequence.

The next example is about projecting onto "ℓ_1-subsets". Before giving it we need a definition. Let X and Y be Banach spaces, and let (x_i) and (y_i) be sequences in X and in Y, respectively (either both infinite, or both finite of the same length). For $C>0$ we say that (x_i) and (y_i) are *C-equivalent*, written $(x_i)\stackrel{C}{\sim}(y_i)$ if there exist constants $A>0$ and $B>0$ with $B/A\leq C$ such that

$$A\Big\|\sum_i a_ix_i\Big\|\leq\Big\|\sum_i a_iy_i\Big\|\leq B\Big\|\sum_i a_ix_i\Big\|$$

for all $(a_i)\in c_{00}$. If only the second inequality holds, then we say (x_i) *B-dominates* (y_i), and write $(y_i)\stackrel{B}{\lesssim}(x_i)$. Let (e_i) be the unit vector basis of ℓ_1. Given a real number $\delta > 0$ and a sequence (x_i) in a Banach space, set $\mathcal{F}\big(\delta,(x_i)\big) = \big\{E \in [\mathbb{N}]^{<\omega} : (x_i)_{i\in E} \stackrel{1/\delta}{\sim} (e_i)_{i=1}^{|E|}\big\}$. In Section 6 we will present a result due to Argyros, Mercourakis, Tsarpalias [5] of which the following is an immediate consequence.

Theorem 1.4. *For each $\delta \in (0,1]$ there exists a constant C such that every normalized, weakly null sequence has a subsequence (x_i) that is $\mathcal{F}\big(\delta,(x_i)\big)$-unconditional with constant C. Moreover, $C \leq 16\log_2\big(1/\delta\big)$ for $\delta < 1/4$.*

As we shall later see, finding the best constant C in the above result is closely related to Problem 1.2. Indeed, if Problem 1.2 has a positive answer, then the above theorem is valid with a constant C not depending on δ. Another problem of interest (although we shall not address it in this paper) is to determine which symmetric bases could replace the unit vector basis of ℓ_1 in the definition of $\mathcal{F}\big(\delta,(x_i)\big)$. We note that projecting onto "c_0-subsets" can always be done: every basic sequence dominates the unit vector basis of c_0. In fact, by Theorem 3.1 below, for every $\epsilon > 0$ every normalized, weakly null sequence has a basic subsequence that $(1+\epsilon)$-dominates the unit vector basis of c_0.

We now describe a different scheme for defining partial unconditionality from the one above. We will denote by $[\mathbb{N}]^{<\omega}$ the set of all finite subsets of $\mathbb{N}$. Let $\mathcal{F}$ be a subset of $c_{00} \times [\mathbb{N}]^{<\omega}$. We say that the sequence (x_i) is *$\mathcal{F}$-unconditional with constant C* if

$$\Big\|\sum_{i\in E} a_i x_i\Big\| \leq C\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \tag{2}$$

holds whenever $\boldsymbol{a} = (a_i) \in c_{00}$, and either $(\boldsymbol{a}, E) \in \mathcal{F}$ or $\boldsymbol{a}$ is arbitrary and E is an initial segment of $\mathbb{N}$. Observe that such a sequence is a basic sequence with constant C, i.e. we can uniformly project onto initial segments with constant C. However, in general, for a given finite set $E \subset \mathbb{N}$ we can only project certain vectors onto E with uniform constant; namely the vectors $\sum_i a_i x_i$ for which the pair $\big((a_i), E\big)$ belongs to $\mathcal{F}$. So this kind of partial unconditionality is of a non-linear nature. Both δ-near-unconditionality and the quasi-greedy property are examples of this. If we let $\mathcal{F}$ to be the set of all pairs $(\boldsymbol{a}, E)$ such that $\boldsymbol{a} = (a_i) \in c_{00}$ and $E = \{i \in \mathbb{N} : |a_i| \geq \delta\}$ for some $\delta > 0$, then (x_i) is $\mathcal{F}$-unconditional if and only if it is quasi-greedy. If for a fixed $\delta \in (0,1)$ we let $\mathcal{F}_\delta$ be the set of pairs $(\boldsymbol{a}, E)$ such that $\boldsymbol{a} = (a_i) \in c_{00}$, $|a_i| \leq 1$ for all $i \in \mathbb{N}$, and $E \subset \{i \in \mathbb{N} : |a_i| \geq \delta\}$, then (x_i) is $\mathcal{F}_\delta$-unconditional if and only if it is δ-near-unconditional.

Problem 1.5. *Does every normalized, weakly null sequence have a quasi-greedy subsequence, or more generally, does every infinite-dimensional Banach space contain a quasi-greedy basic sequence?*

Dilworth, Kalton and Kutzarova [10, Theorem 5.4] proved that if a normalized, weakly null sequence (x_i) has a spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon > 0$ there is a quasi-greedy subsequence of (x_i) with constant $3+\epsilon$. This is not too surprising: if we are in some sense far from c_0, then we expect a uniform bound on the number of large coefficients in a norm-1 vector, from which the result follows by Schreier-unconditionality. We shall spell out this argument later which will also show (using a version of Schreier-unconditionality, Theorem 3.1 below) that if (x_i) is a normalized, weakly null sequence with spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon > 0$ and for any $\delta \in (0, 1)$ there is a δ-near-unconditional subsequence of (x_i) with constant $1+\epsilon$.

Thus Problems 1.2 and 1.5 have positive answers if we are "far" from c_0. However, they are still open in general. What we do know is that one cannot hope to find for any $\epsilon > 0$ subsequences of normalized, weakly null sequences that are δ-near-unconditional or quasi-greedy with constant $1 + \epsilon$. We are going to prove this in Section 8 (Example 8.7). We will also show in Section 3 that a positive answer to Problem 1.2 implies a positive answer to Problem 1.5.

One could be forgiven for thinking that a positive answer to Problem 1.2 would easily imply that every normalized, weakly null sequence has an unconditional subsequence. It is certainly true that in a δ-near-unconditional sequence we can project onto any *subset* of the co-ordinates with 'large' coefficients (unlike in a quasi-greedy sequence). However, there are two restrictions. First, there is a normalization: $|a_i| \leq 1$ for all $i \in \mathbb{N}$ whenever $(\boldsymbol{a}, E) \in \mathcal{F}_\delta$ (where $\mathcal{F}_\delta$ is defined just before the statement of Problem 1.5). Without this condition, for any pair $(\boldsymbol{a}, E)$, there would exist a positive real number r such that $(r\boldsymbol{a}, E) \in \mathcal{F}_\delta$, and hence a δ-near-unconditional sequence would indeed be unconditional. Second, even if there is a constant K such that $K(\delta) < K$ for all $\delta > 0$, the subsequence that is δ-near-unconditional with constant K, and that we can find in a given normalized, weakly null sequence may very well depend on δ. In other words there is no obvious reason why a positive answer to Problem 1.2 would find, in every normalized weakly null sequence, a subsequence that is δ-near-unconditional with constant K for *all* $\delta > 0$ (which again would be unconditional). Note that the standard diagonal argument would give a subsequence that is δ-near-unconditional with constant $N(\delta)+2K$ for *all* $\delta > 0$, where N is an integer-valued function with $\lim_{\delta\to 0} N(\delta) = \infty$.

This paper will be organized as follows. In the next section we introduce the concept of a bounded-oscillation-unconditional basic sequence, which is a new type of partial unconditionality. We then prove our main result (Theorem 2.1) that states that every normalized, weakly null sequence has a bounded-oscillation-unconditional subsequence. The combinatorial machinery that we set up in order to prove our main result will be subsequently employed to prove other partial unconditionality results. We will use it in Section 3 to give a new proof of Schreier unconditionality. Here we will also deduce Elton's theorem from our main result, Theorem 2.1. We will then prove that a positive answer to Problem 1.2 implies a positive answer to Problem 1.5. When we are "far" from c_0, these problems do have positive solutions as mentioned above. We present a proof of this fact at the end of Section 3.

In Section 4 we introduce various constants similar to the constant $K(\delta)$ defined above. These will allow us to quantify the relationships between various notions of partial unconditionality. We will also show that for solving Problem 1.2 one can restrict attention to the Banach spaces of continuous functions on countable, compact, Hausdorff spaces. In Section 5 we raise the question whether there is a uniform constant C such that every sequence equivalent to the unit vector basis of c_0 has an unconditional subsequence with constant C. This turns out to be closely related to Problem 1.2. The proof will again use our combinatorial machinery.

In the following two sections we revisit convex unconditionality of Argyros, Mercourakis, Tsarpalias [5], and unconditionality of certain sequences in spaces of continuous functions. Using our approach we give new proofs of known results and establish a duality between them and near-unconditionality. One small advantage is that we obtain, in some cases, better constants (although, these can also be obtained by very simple modifications of the original proofs). More importantly, connecting many different results by a unified approach and by establishing equivalences between them, we increase the possibility of tackling Problem 1.2 and hence obtain solutions to many other problems. The use of our combinatorial machinery also suggests a way of producing a counterexample in case Problem 1.2 has a negative answer. This is discussed in the last section of the paper.

In the final section we will have a closer look at our combinatorial machinery. We give a necessary and sufficient condition for a positive answer to Problem 1.2 (c.f. Proposition 8.1). To decide if this condition can be satisfied in general one is lead to consider certain combinatorial data attached to subsequences of a normalized, weakly null sequence. We will study this data on its own right as a purely combinatorial object. Our results will be used at the end to give an example that among other things shows that $\sup_{\delta>0} K(\delta)$ is strictly greater than 1.

2. Main results

Given a sequence $\boldsymbol{a}=(a_i)$ of real numbers, we define its *support* to be the set $\mathrm{supp}(\boldsymbol{a})=\{i\in\mathbb{N}:\ a_i\neq 0\}$. If this set is finite we call $\boldsymbol{a}$ *finitely supported*. Recall that c_{00} denotes the space of finitely supported sequences of real numbers. Given $\boldsymbol{a}=(a_i)\in\mathrm{c}_{00}$ and a subset E of $\mathbb{N}$ we define the *oscillation* $\mathrm{osc}(\boldsymbol{a},E)$ *of* $\boldsymbol{a}$ *over* E as

$$\mathrm{osc}(\boldsymbol{a},E)=\sup\left\{\frac{|a_i|}{|a_j|}\,:\,i,j\in E,\ a_j\neq 0\right\}.$$

For subsets E and F of $\mathbb{N}$ we write $E<F$ if $m<n$ for all $m\in E$ and for all $n\in F$. We say that a sequence $E_1,\dots,E_n$ of subsets of $\mathbb{N}$ is *successive* if $E_1<\ldots<E_n$. A decomposition $E=\bigcup_{j=1}^n E_j$ of a finite set E will be called a *Schreier decomposition* if $E_1<\ldots<E_n$ is a successive sequence of non-empty sets such that $n\leq\min E_1$, i.e. the set $\{\min E_1\ldots\min E_n\}$ belongs to $\mathcal{S}_1$.

We now come to the main definition. Let $C,D,d\in[1,\infty)$. We say that a basic sequence (x_i) in a Banach space X is (D,d)*-bounded-oscillation-unconditional with constant* C if for every $\boldsymbol{a}=(a_i)\in\mathrm{c}_{00}$, and for every finite set $E\subset\mathbb{N}$ with $\mathrm{osc}(\boldsymbol{a},E)\leq D$, we have

$$\Big\|\sum_{i\in E}a_ix_i\Big\|\leq C\Big\|\sum_{i=1}^{\infty}a_ix_i\Big\|$$

provided E has a Schreier decomposition $E=\bigcup_{j=1}^n E_j$ such that $\mathrm{osc}(\boldsymbol{a},E_j)\leq d$ for each $j=1,\dots,n$. Note that without this proviso the sequence (x_i) would be a $1/D$-near-unconditional sequence.

Our main theorem is the following.

Theorem 2.1. *For all $d\in[1,\infty)$, there is a constant $C\leq 8d$ such that for all $D\in[1,\infty)$ and for any $\epsilon>0$ every normalized, weakly null sequence has a subsequence that is a (D,d)-bounded-oscillation-unconditional basic sequence with constant $C+\epsilon$.*

Note that if $\boldsymbol{a}=(a_i)\in\mathrm{c}_{00},\ E\in[\mathbb{N}]^{<\omega}$ and $\mathrm{osc}(\boldsymbol{a},E)\leq D$, then we can write E as the disjoint union of $n\leq\lfloor\log_2(D)\rfloor+1$ sets $E_1,\dots,E_n$ such that $\mathrm{osc}(\boldsymbol{a},E_j)\leq 2$ for each $j=1,\dots,n$. So without the assumption that the sets in a Schreier decomposition are successive the above result would be a positive answer to Problem 1.2.

A key ingredient in the proof of Theorem 2.1 is a purely combinatorial result which we call the Matching Lemma (Theorem 2.2). In its proof and in much of this paper we will be making heavy use of infinite Ramsey theory. For this reason we now recall some notation and results from the subject. For a subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M and by $[M]^{\omega}$

the set of all infinite subsets of M. The power-set $2^{\mathbb{N}}$ of $\mathbb{N}$ is equipped with the product topology, and all subspaces will carry the subspace topology. A collection $\mathcal{U}\subset[\mathbb{N}]^{\omega}$ is said to be *Ramsey* if for all $L\in[\mathbb{N}]^{\omega}$ there exists $M\in[L]^{\omega}$ such that either $[M]^{\omega}\subset\mathcal{U}$ or $[M]^{\omega}\subset\mathcal{U}^{\complement}$, where $\mathcal{U}^{\complement}=[\mathbb{N}]^{\omega}\setminus\mathcal{U}$ denotes the complement of $\mathcal{U}$. One example of an infinite Ramsey theorem, due to Galvin and Prikry [11], states that every Borel subset $\mathcal{U}$ of $[\mathbb{N}]^{\omega}$ is Ramsey. More generally, whenever $[\mathbb{N}]^{\omega}$ is partitioned into finitely many Borel sets, every infinite subset L of $\mathbb{N}$ has an infinite subset M such that $[M]^{\omega}$ is contained in one of the Borel sets of the partition. The strongest result of this type was proved by Ellentuck [8]; his result concerns topological characterizations of Ramsey sets. In all our applications (and indeed in most applications to Banach space theory) it will suffice to know that open sets (and hence closed sets) are Ramsey. This was first proved by Nash-Williams [19]. Following tradition we will often talk about colourings instead of partitions. This and other pieces of terminology will be introduced as we go along. For a very good introduction to infinite Ramsey theory see [7]. An extensive account is presented in [13].

Because of the importance of Theorem 2.2 we shall write out the proof in detail including the fairly technical parts that could otherwise be left as exercise. There will also be several remarks pointing out the main ideas of the argument.

We need one final piece of notation before stating Theorem 2.2. For subsets A,B of $\mathbb{N}$ we write $A\prec B$ if A is an initial segment of B.

Theorem 2.2 (Matching Lemma)**.** *Let $n\in\mathbb{N}$. Assume that for every infinite subset M of $\mathbb{N}$ we are given a successive sequence*

$$F_1^M<\cdots<F_n^M$$

of non-empty, finite subsets of M. Further assume that for each $j=1,\dots,n$ the function $F_j\colon[\mathbb{N}]^{\omega}\to[\mathbb{N}]^{<\omega}$, $M\mapsto F_j^M$, is continuous. Then for all $N\in[\mathbb{N}]^{\omega}$ there exist $L,M\in[N]^{\omega}$ such that

(i) for each $j=1,\dots,n$ either $F_j^L\prec F_j^M$ or $F_j^M\prec F_j^L$, and

(ii) $L\cap M=\bigcup_{j=1}^{n}F_j^L\cap F_j^M$.

Proof. We begin by setting up some notation. Let $F_L=\bigcup_{j=1}^{n}F_j^L$ for each $L\in[\mathbb{N}]^{\omega}$. We are going to define a *finite colouring* c of pairs (L,l), where L is an infinite subset of $\mathbb{N}$ and $l\in L$. In other words we are going to define a function c on the set of all such pairs taking values in some finite set whose elements will be referred to as *colours*. So fix $L\in[\mathbb{N}]^{\omega}$ and $l\in L$. If $l\in F_i^L$ for some $i=1,\dots,n$, then we set $c(L,l)=i$. If $l\notin F_L$, and the minimum of $\{l'\in L: l'>l\}\cap F_L$ belongs to F_i^L, then we set $c(L,l)=i+$. Finally, if $l>\max F_L$, then we set $c(L,l)=+$. Clearly there exists $l_0\in L$ with $c(L,l_0)=+$, and for such an l_0 we have $c(L,l)=+$ for all $l\in L$ with $l\geq l_0$.

We now prove a preliminary result.

Claim. For all pairs (F,X), where $F\in[\mathbb{N}]^{<\omega}$ and $X\in[\mathbb{N}]^{\omega}$, there exist $Y\in[X]^{\omega}$ and a colour λ such that $F<Y$ and $c(F\cup V,\min V)=\lambda$ for all $V\in[Y]^{\omega}$.

This result says that we can *predict* the colour of the next point, i.e. no matter how we extend F to an infinite set, the colour of the next point will always be the same (provided the extension is inside Y).

To see the Claim define a finite colouring d of $[\mathbb{N}]^{\omega}$ by setting $d(V)=c(F\cup V,\min V)$ for every $V\in[\mathbb{N}]^{\omega}$. It follows from the continuity of the maps F_j that if λ is a colour other than $+$, the corresponding *colour-class*, i.e. the collection $\{V\in[\mathbb{N}]^{\omega}: d(V)=\lambda\}$ is an open subset of $[\mathbb{N}]^{\omega}$. It follows that the colour-class of $+$ is closed. Since open sets and closed sets are Ramsey, it follows that there is an infinite subset Y of X all whose infinite subsets have the same colour. Replacing Y by a smaller set if necessary we may clearly assume that $F<Y$.

We now turn to the proof of Theorem 2.2. Fix $N\in[\mathbb{N}]^{\omega}$. We shall build infinite subsets L and M of N from recursively constructed sequences $l_1\leq l_2\leq\dots,\ m_1\leq m_2\leq\dots$ of positive integers in N. Along the way we shall also construct a sequence $P_0\supset P_1\supset P_2\supset\dots$ of infinite subsets of N, and sequences $(\lambda_k)_{k=0}^{\infty}$ and $(\mu_k)_{k=0}^{\infty}$ of colours. To do this we shall repeatedly apply the Claim to predict the colour of the next points in L and M. Depending on these colours, we either *match* or *disjointify* (i.e. we choose the next points of L and M to be the same or distinct, respectively) in a way to be explained.

To start the construction apply the Claim with $F=\emptyset$ and $X=N$. This yields an infinite subset Y of X and a colour λ such that $c(V,\min V)=\lambda$ for all $V\in[Y]^{\omega}$. Let us set $P_0=Y$ and $\lambda_0=\mu_0=\lambda$.

For the recursive step suppose that $k\geq 0$ and that $l_r,\ m_r$ for $1\leq r\leq k$ and $P_r,\ \lambda_r,\ \mu_r$ for $0\leq r\leq k$ have been chosen. We also assume that setting $A_k=\{l_r: 1\leq r\leq k\}$ and $B_k=\{m_r: 1\leq r\leq k\}$ the following hold.

(3) $\quad A_k<P_k \quad\text{and}\quad B_k<P_k,$

(4) $\quad c(A_k\cup Q,\min Q)=\lambda_k,\ c(B_k\cup Q,\min Q)=\mu_k \quad\text{for all}\quad Q\in[P_k]^{\omega}.$

Note that when $k=0$ these assumptions are satisfied by the choice of P_0. To choose l_{k+1} and m_{k+1} we consider four cases (to be followed by an explanation of each case).

Case 1. If $\lambda_k = \mu_k = i$ for some $i \in \{1, \dots, n\}$, then we choose $l_{k+1} = m_{k+1}$ to be an arbitrary element of P_k.

Case 2. If one of

(a) neither λ_k nor μ_k belongs to $\{1, \dots, n\}$,

(b) $\{\lambda_k, \mu_k\} = \{i, j+\}$ for some $1 \le i < j$, or

(c) at least one of λ_k and μ_k is $+$

holds, then we choose l_{k+1} and m_{k+1} to be distinct elements of P_k.

Case 3. If $\lambda_k = i$ and either $\mu_k = j$ for some $1 \le j < i$ or $\mu_k = j+$ for some $1 \le j \le i$, then we set $l_{k+1} = l_k$ and choose m_{k+1} to be an arbitrary element of P_k.

Case 4. If $\mu_k = i$ and either $\lambda_k = j$ for some $1 \le j < i$ or $\lambda_k = j+$ for some $1 \le j \le i$, then we set $m_{k+1} = m_k$ and choose l_{k+1} to be an arbitrary element of P_k.

In Case 1 the predicted colours λ_k and μ_k are both i, i.e. the next points of L and M belong to F_i^L and F_i^M, respectively. In order to satisfy (i) in the statement of the theorem we need to match, i.e. we need to choose the next points of L and M to be the same. In Case 3 the predicted colours, λ_k and μ_k tell us that the next point of L will belong to F_i^L, whereas the next point of M will not be in F_i^M but there will be later points of M that do belong to F_i^M. So to satisfy (i) we put the construction of L on hold (we don't choose the next point of L yet). Case 4 is similar. Finally, Case 2 covers the remaining possibilities, when in order to satisfy (ii) we disjointify, i.e. choose the next points of L and M to be distinct.

Note that when $k = 0$ only Cases 1 and 2 can arise, since $\lambda_0 = \mu_0$. When $k \ge 1$ we have $l_k \le l_{k+1}$ and $m_k \le m_{k+1}$ in all the cases, as required. Let us at this point set $l_0 = m_0 = 0$ in order to avoid having to consider the first step of the construction separately from the recursive steps. Observe that for any $k \ge 0$, if $l_{k+1} > l_k$, then $l_{k+1} \in P_k$. Similarly, if $m_{k+1} > m_k$, then $m_{k+1} \in P_k$.

To complete the recursive step we need to choose P_{k+1}, λ_{k+1} and μ_{k+1}. First set $A_{k+1} = A_k \cup \{l_{k+1}\}$ and $B_{k+1} = B_k \cup \{m_{k+1}\}$. Then apply the Claim with $F = A_{k+1}$ and $X = P_k$ to obtain an infinite subset $\tilde{P}$ of P_k and a colour λ_{k+1} such that $A_{k+1} < \tilde{P}$ and $c(A_{k+1} \cup Q, \min Q) = \lambda_{k+1}$ for all $Q \in [\tilde{P}]^\omega$. Now apply the Claim again with $F = B_{k+1}$ and $X = \tilde{P}$ to obtain an infinite subset P_{k+1} of $\tilde{P}$ and a colour μ_{k+1} such that $B_{k+1} < P_{k+1}$ and $c(B_{k+1} \cup Q, \min Q) = \mu_{k+1}$ for all $Q \in [P_{k+1}]^\omega$. With these choices it is clear that the assumptions for the next recursive step (i.e. (3) and (4) with k replaced by $k+1$) are satisfied. Observe that if $l_{k+1} = l_k$, then $\lambda_{k+1} = \lambda_k$, and if $m_{k+1} = m_k$, then $\mu_{k+1} = \mu_k$.

Having completed the recursive construction let us put $L = \{l_r : r \in \mathbb{N}\}$ and $M = \{m_r : r \in \mathbb{N}\}$. Notice that for any $k \ge 0$, if $l_{k+1} > l_k$, then $L_k = \{l_r :$

$r > k\}$ is a subset of P_k. Indeed, for $r > k$ we have $l_r = l_{s+1} > l_s$ for some s with $k \le s < r$, and hence $l_r \in P_s \subset P_k$. So if in addition L is infinite, then $c(L, l_{k+1}) = c(A_k \cup L_k, \min L_k) = \lambda_k$. Similarly, for any $k \ge 0$ if $m_{k+1} > m_k$, then $M_k = \{m_r : r > k\}$ is a subset of P_k, and if in addition M is infinite, then we have $c(M, m_{k+1}) = c(B_k \cup M_k, \min M_k) = \mu_k$.

We will now verify that L and M are indeed infinite sets. We argue by contradiction. Assume, for example, that L is finite. Then for some $k_0 \in \mathbb{N}$ we have $l_{k+1} = l_k$ for all $k \ge k_0$. It follows that for every $k \ge k_0$ we applied Case 3 in the k^{th} step of the recursion. Hence for every $k \ge k_0$ we have $m_{k+1} > m_k$ and $c(M, m_{k+1}) = \mu_k \ne +$. This contradiction shows that L is infinite. Similar reasoning gives that M must also be infinite.

Next let us fix $i \in \{1, \dots, n\}$. We need to show that either $F_i^L \prec F_i^M$ or $F_i^M \prec F_i^L$. We argue by contradiction. Suppose that there exist $l \in F_i^L$ and $m \in F_i^M$ such that

$$l \ne m \quad \text{and} \quad \{l' \in F_i^L : l' < l\} = \{m' \in F_i^M : m' < m\}. \tag{5}$$

For some $k \ge 0$ and $k' \ge 0$ we have $l = l_{k+1} > l_k$ and $m = m_{k'+1} > m_{k'}$. Then $\lambda_k = c(L, l) = i$ and $\mu_{k'} = c(M, m) = i$. From now on assume that $k \le k'$ (the case $k \ge k'$ is similar). There exists k'' with $k \le k'' \le k'$ such that $m_k = m_{k''} < m_{k''+1}$, and so $\mu_k = \mu_{k''} = c(M, m_{k''+1})$. From $m_{k''+1} \le m$ and $c(M, m) = i$ we deduce that the colour μ_k is either j or $j+$ for some j with $1 \le j \le i$. Hence in the k^{th} recursive step we applied either Case 1 or Case 3. Case 1 leads to $l = l_{k+1} = m_{k+1} \in F_i^M$ and $l \le m$ which contradicts (5), whereas Case 3 gives $l_{k+1} = l_k$ contradicting the choice of k.

We are left to show that $L \cap M \subset \bigcup_{i=1}^n F_i^L \cap F_i^M$ (the reverse inclusion being obvious). Let l belong to $L \cap M$. There exist $k \ge 0$ and $k' \ge 0$ such that $l = l_{k+1} = m_{k'+1}$ and $l_{k+1} > l_k$, $m_{k'+1} > m_{k'}$. Then $l \in P_k \setminus P_{k+1}$ and $l \in P_{k'} \setminus P_{k'+1}$, from which we get $k = k'$. So we have $l = l_{k+1} = m_{k+1}$ and $l_{k+1} > l_k$, $m_{k+1} > m_k$. It follows immediately that in the k^{th} step of the recursion we must have been in Case 1. Hence for some $i = 1, \dots, n$ we have $c(L, l) = \lambda_k = i$ and $c(M, l) = \mu_k = i$, i.e. $l \in F_i^L \cap F_i^M$, as required. This completes the proof of Theorem 2.2. $\square$

Some minor modifications of the proof and a simple diagonalization procedure yields a corollary that we shall refer to as the Schreier version of the Matching Lemma. The diagonalization process will be used later on, so we state it separately as an abstract principle. A family $\mathcal{A}$ of finite subsets of $\mathbb{N}$ is *thin* if no element of $\mathcal{A}$ is the proper initial segment of another element of $\mathcal{A}$. The following result was proved by Nash-Williams [19]: if a thin family $\mathcal{A}$ is finitely coloured, then for all $L \in [\mathbb{N}]^\omega$ there exists $M \in [L]^\omega$ such that $[M]^{<\omega} \cap \mathcal{A}$ is monochromatic. To see this, simply give an infinite set L the colour of its unique initial segment in $\mathcal{A}$ (introducing a new colour for infinite sets with no initial segment in $\mathcal{A}$). Clearly, each colour-class is either open or closed, so the result follows. An easy diagonalization argument then gives the following result. (A much stronger statement is given by Pudlák and Rödl [23].)

Proposition 2.3. *Let $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ be a thin family. For each $k \in \mathbb{N}$ let S_k be a finite set, and let $c\colon \mathcal{A} \to \bigcup_{k=1}^{\infty} S_k$ be a colouring of $\mathcal{A}$ so that for all $F \in \mathcal{A}$ we have $c(F) \in S_k$, where $k = \min F$. Then for all $L \in [\mathbb{N}]^\omega$ there exists $M \in [L]^\omega$ such that if $A, B \in [M]^{<\omega} \cap \mathcal{A}$ and $\min A = \min B$, then $c(A) = c(B)$.* $\square$

We are now ready to state and prove the promised corollary to (the proof of) Theorem 2.2.

Corollary 2.4 (Schreier version of the Matching Lemma)**.** *Assume that for each $M \in [\mathbb{N}]^\omega$ we have a positive integer n_M and non-empty finite subsets A_M, $F_1^M < \dots < F_{n_M}^M$ of M such that*

$$\bigcup_{j=1}^{n_M} F_j^M \subset A_M \qquad \text{and} \qquad n_M \le \min F_1^M = \min A_M .$$

Further assume that the function $M \mapsto A_M \colon [\mathbb{N}]^\omega \to [\mathbb{N}]^{<\omega}$ is continuous, that the family $\mathcal{A} = \{A_M : M \in [\mathbb{N}]^\omega\}$ is thin, and that for all $L, M \in [\mathbb{N}]^\omega$ if $A_L = A_M$, then $n_L = n_M$ and $F_j^L = F_j^M$ for each $j = 1, \dots, n_L$. Then for all $N \in [\mathbb{N}]^\omega$ there exists $L, M \in [N]^\omega$ with $n_L = n_M$ such that

(i) for each $j = 1, \dots, n_L$ either $F_j^L \prec F_j^M$, or $F_j^M \prec F_j^L$, and

(ii) $L \cap M = \bigcup_{j=1}^{n_L} F_j^L \cap F_j^M$.

Proof. We first define a colouring of $\mathcal{A}$ by giving each A_M, $M \in [\mathbb{N}]^\omega$, the colour n_M. This is well-defined by the assumptions. By Proposition 2.3 there exists $N_1 \in [N]^\omega$ such that for all $L, M \in [N_1]^\omega$ if $\min F_1^L = \min F_1^M$, then $n_L = n_M$.

We now follow the proof of Theorem 2.2. We define the colouring c on pairs (L, l) as before. Although this time c is a possibly infinite colouring, the colouring d used in the proof of the Claim is finite, so the Claim remains valid. We then carry out the recursive construction that produces the sets L and M. The only changes we need is to work inside N_1 (rather than N), and to replace in Cases 1–4 each occurence of $\{1, \dots, n\}$ by $\mathbb{N}$. The verification that L and M are infinite is the same as before.

At this point we need to insert the observation that $\min F_1^L = \min F_1^M$. To see this choose $k \geq 0$ and $k' \geq 0$ such that $\min F_1^L = l_{k+1} > l_k$ and $\min F_1^M = m_{k'+1} > m_{k'}$ so that $\lambda_k = \mu_{k'} = 1$. Assume that $k \leq k'$ (the case $k' \leq k$ is similar). Then $m_k \leq m_{k'} < m_{k'+1}$, and so μ_k is either 1 or 1+. It follows that in the k^{th} step of the recursion we were either in Case 1, in which case we have $m_{k+1} = l_{k+1}$ (and $k = k'$), as required, or we were in Case 3, in which case we obtain $l_k = l_{k+1}$, which contradicts the choice of k.

We now have $n_L = n_M$ by our initial application of Proposition 2.3. To finish the proof we verify properties (i) and (ii) exactly as in the proof Theorem 2.2 (letting n in the proof stand for n_L). $\square$

Applications of the Matching Lemma and of its Schreier version will require two further lemmas. To motivate the first one of these we now give a preview of the type of argument that will follow. Consider the general problem of starting with a normalized, weakly null sequence (x_i) and seeking a subsequence with a certain desired property. Arguing by contradiction, we assume that for all $M \in [\mathbb{N}]^\omega$ we have a witness w_M to the lack of the desired property in the subsequence $(x_i)_{i \in M}$. The witness w_M will then give rise in a very natural way to finitely many subsets $F_1^M < F_2^M < \dots$ of M. Lemma 2.5 below will allow us to choose w_M from the set of all possible witnesses for M in a "continuous" way so that among other things the assumptions of the Matching Lemma or its corollary are satisfied. In typical examples a witness w_M has as a constituent part some functional x_M^*. A priori we will not be able to assume that the support of x_M^*, i.e. the set $\operatorname{supp}(x_M^*) = \{i \in \mathbb{N} : x_M^*(x_i) \neq 0\}$ is contained in M, precisely because we lack unconditionality. In Lemma 2.6 we show that we can stabilize, i.e. we can pass to some infinite set with respect to which the property $\operatorname{supp}(x_M^*) \subset M$ can be assumed (provided the choice of x_M^* had already been made in a "continuous" manner).

Lemma 2.5. *Let $\Omega = \bigcup_{r=1}^\infty \Omega_r$ be an arbitrary set written as the union of a countably infinite collection of its subsets. Let*

$$\Phi \colon [\mathbb{N}]^\omega \to 2^\Omega \setminus \{\emptyset\}$$

be a function into the set of non-empty subsets of Ω. Assume that for all $r \in \mathbb{N}$ and for all $L, M \in [\mathbb{N}]^\omega$ we have

$$L \cap \{1, \dots, r\} = M \cap \{1, \dots, r\} \quad \Longrightarrow \quad \Phi(L) \cap \Omega_r = \Phi(M) \cap \Omega_r.$$

Then there is a function

$$\phi \colon [\mathbb{N}]^\omega \to \Omega$$

such that

(i) $\phi(M)\in\Phi(M)$ *for all* $M\in[\mathbb{N}]^\omega$, *and*

(ii) ϕ *is continuous if* Ω *is given the discrete topology.*

Proof. Fix a well-ordering of Ω. For $M\in[\mathbb{N}]^\omega$ let

$$r(M)=\min\{r\in\mathbb{N}:\ \Phi(M)\cap\Omega_r\neq\emptyset\}.$$

Define $\phi(M)$ to be the least element of $\Phi(M)\cap\Omega_{r(M)}$ in our chosen well-ordering. We claim that $\phi\colon[\mathbb{N}]^\omega\to\Omega$ has the required properties.

Clearly $\phi(M)\in\Phi(M)$ for all infinite subsets M of $\mathbb{N}$. To verify continuity fix $M\in[\mathbb{N}]^\omega$ and set $r=r(M)$. Let $[r]=\{1,\dots,r\}$. If $L\in[\mathbb{N}]^\omega$ satisfies $L\cap[r]=M\cap[r]$, then for each $1\le r'\le r$ we have $\Phi(L)\cap\Omega_{r'}=\Phi(M)\cap\Omega_{r'}$, which is the empty set for $r'<r$ and is not empty for $r'=r$. It follows that $r(L)=r(M)$, which in turn implies that $\phi(L)=\phi(M)$. This shows that ϕ maps the neighbourhood $\{L\in[\mathbb{N}]^\omega:\ L\cap[r]=M\cap[r]\}$ of M onto $\phi(M)$. $\square$

Lemma 2.6. *Let* c_0 *be equipped with the topology of pointwise convergence on* $\mathbb{N}$. *Let* $f\colon[\mathbb{N}]^\omega\to c_0,\ M\mapsto f_M$, *be a continuous function such that every sequence in the image of* f *has a cluster point in* c_0. *Then for every* $\epsilon>0$ *and for every* $M\in[\mathbb{N}]^\omega$ *there exists* $N\in[M]^\omega$ *such that for all* $P\in[N]^\omega$ *we have*

$$\sum_{i\in N\setminus P}|f_P(i)|\le\epsilon,$$

i.e. the support $\mathrm{supp}(f_P)=\{i\in\mathbb{N}:\ f_P(i)\neq0\}$ *of* f_P *relative to the set* N *is contained in* P *up to a small perturbation.*

Proof. For $L\in[\mathbb{N}]^\omega$ let us write L' as a temporary notation for $L\setminus\{\min L\}$. For $F\in[\mathbb{N}]^{<\omega}$ and $\delta>0$ let $\mathcal{U}_{F,\delta}$ be the collection of all infinite subsets L of $\mathbb{N}$ for which we have

$$|f_{F\cup L'}(\min L)|<\delta.$$

As a preliminary step we first prove the following claim. Given $F\in[\mathbb{N}]^{<\omega}$ and $L\in[\mathbb{N}]^\omega$, there exists $\tilde{L}\in[L]^\omega$ such that $[\tilde{L}]^\omega\subset\mathcal{U}_{F,\delta}$. Indeed, the continuity of f implies that $\mathcal{U}_{F,\delta}$ is an open set, and hence it is Ramsey. Thus there exists $\tilde{L}\in[L]^\omega$ such that either $[\tilde{L}]^\omega\subset\mathcal{U}_{F,\delta}$ or $[\tilde{L}]^\omega\subset\mathcal{U}^{\complement}_{F,\delta}$. So to prove the claim we need to exclude the second alternative. We argue by contradiction. Assume that $[\tilde{L}]^\omega\subset\mathcal{U}^{\complement}_{F,\delta}$. Let $l_1<l_2<\dots$ be an enumeration of $\tilde{L}$, and for $n\in\mathbb{N}$ let $L_n=\{l_i:\ i>n\}$. Then $L_n\cup\{l_i\}\in\mathcal{U}^{\complement}_{F,\delta}$, and hence

$$|f_{F\cup L_n}(l_i)|\ge\delta\quad\text{whenever }1\le i\le n.$$

Let $x\in c_0$ be a cluster point of the sequence $(f_{F\cup L_n})_{n=1}^\infty$. From the above we have $|x(l_i)|\geq\delta$ for all $i\in\mathbb{N}$ contradicting that x is an element of c_0. This completes the proof of the claim.

To prove Lemma 2.6 let us fix $\epsilon>0$ and $M\in[\mathbb{N}]^\omega$. Choose real numbers $\epsilon_i>0,\ i=1,2,\dots$, such that $\sum_{i=1}^\infty\epsilon_i<\epsilon$. We shall now recursively construct a sequence $n_1<n_2<\dots$ of positive integers, and a sequence $L_0\supset L_1\supset L_2\supset\dots$ of infinite subsets of $\mathbb{N}$ as follows. To start with, set $L_0=M$. Assume that for some $k\geq 1$ we have chosen n_i for $1\leq i<k$ and L_i for $0\leq i<k$. Let $F_1,\dots,F_K$ be an enumeration of the power-set of $\{n_1,\dots,n_{k-1}\}$. Then choose a chain $L_{k-1}=\tilde{L}_0\supset\tilde{L}_1\supset\dots\supset\tilde{L}_K$ of infinite sets such that for each $j=1,\dots,K$ we have $[\tilde{L}_j]^\omega\subset\mathcal{U}_{F_j,\epsilon_k}$. This can be done by our preliminary claim. Now set $n_k=\min\tilde{L}_K$ and $L_k=\tilde{L}_K\setminus\{n_k\}$. Note that $n_k\in L_{k-1},\ L_k\subset L_{k-1},\ n_k<L_k$ and

$$|f_{F\cup Q}(n_k)|<\epsilon_k \qquad \text{for all } F\subset\{n_1,\dots,n_{k-1}\},\ \text{and } Q\in[L_k]^\omega. \tag{6}$$

Having completed the recursive construction, set $N=\{n_1,n_2,\dots\}$. It is clear that $N\in[M]^\omega$. Given any $P\in[N]^\omega$, if $k\in\mathbb{N}$ with $n_k\notin P$, then $P=F\cup Q$, where

$$F=P\cap\{n_1,\dots,n_{k-1}\},\ \text{and } Q=P\setminus F\in[L_k]^\omega.$$

Hence from (6) we have $|f_P(n_k)|<\epsilon_k$. It follows that

$$\sum_{n\in N\setminus P}|f_P(n)|<\sum_{k=1}^\infty\epsilon_k<\epsilon,$$

as required. $\square$

We are now ready to present a proof for Theorem 2.1. It will be convenient to use the following definition of an ϵ-net F for a subset S of $\mathbb{R}^d$, where $\epsilon>0$ and $d\in\mathbb{N}$: for every $(\alpha_j)_{j=1}^d\in S$ there exists $(\beta_j)_{j=1}^d\in F$ such that $\beta_j\leq\alpha_j\leq\beta_j+\epsilon$ for each $j=1,\dots,d$.

Proof of Theorem 2.1. Fix $C,D,d\in[1,\infty)$. Assume that (x_i) is a normalized, weakly null sequence no subsequence of which is (D,d)-bounded-oscillation-unconditional basic sequence with constant C. We shall deduce that $C\leq 8d$. Fix $\epsilon\in(0,1)$ and then choose an increasing function $\gamma\colon\mathbb{N}\to\mathbb{N}$ such that $\lim_{k\to\infty}\gamma(k)=\infty$ and

$$\gamma(k)+Dk\leq(1+\epsilon)\gamma(k-1) \qquad \text{for all } k\geq 2. \tag{7}$$

For example, we can take $\gamma(k)=k^2$ for $k\geq k_0$ and $\gamma(k)=k_0^2$ for $k<k_0$, where k_0 is sufficiently large.

After passing to a subsequence we may assume that (x_i) is a basic sequence with constant $1+\epsilon$. Then in particular for all $\boldsymbol{a}=(a_i)\in c_{00}$ we have

$$\|\boldsymbol{a}\|_{\ell_\infty} \leq 2(1+\epsilon)\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \leq 4\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\|. \tag{8}$$

We now show that for every infinite subset M of $\mathbb{N}$ there exists a triple $(\boldsymbol{a}, x^*, F)$, which we shall call a *witness for* M, with the following properties.

(9) $\boldsymbol{a} = (a_i)\in c_{00}$, $x^*\in B_{X^*}$ and $F\in[\mathbb{N}]^{<\omega}$;

(10) $F\subset A\subset M$ and $\min F=\min A$, where $A = \operatorname{supp}(\boldsymbol{a})$;

(11) F has a Schreier decomposition $F = \displaystyle\bigcup_{j=1}^{n} F_j$ such that

$\operatorname{osc}(\boldsymbol{a}, F_j) \leq d$ for each $j = 1,\dots,n$;

(12) $1 \leq a_i \leq D$ and $x^*(x_i) > 0$ for all $i \in F$;

(13) $\dfrac{C}{2(1+\epsilon)(2+\epsilon)}\|x\| < \displaystyle\sum_{i\in F} a_i x^*(x_i) \leq \gamma(k) + D$, where

$k = \min F$, and $x = \displaystyle\sum_{i\in M} a_i x_i$.

To see this let us fix $M\in[\mathbb{N}]^{\omega}$. Since $(x_i)_{i\in M}$ is not (D,d)-bounded-oscillation-unconditional with constant C, there exist $\boldsymbol{b}=(b_i)\in c_{00}$ with $\operatorname{supp}(\boldsymbol{b})\subset M$, and a finite subset E of M with a Schreier decomposition $E = \bigcup_{j=1}^{n} E_j$ such that $\operatorname{osc}(\boldsymbol{b}, E)\leq D$, $\operatorname{osc}(\boldsymbol{b}, E_j)\leq d$ for each $j=1,\dots,n$, and

$$\Big\|\sum_{i\in E} b_i x_i\Big\| > C\|y\|,$$

where $y=\sum_{i\in M} b_i x_i$. We may then choose $x^*\in B_{X^*}$ such that

$$\sum_{i\in E} b_i x^*(x_i) > C\|y\|.$$

Replacing $\boldsymbol{b}$ and x^* by $-\boldsymbol{b}$ and $-x^*$ if necessary, we may assume that if we let $E'=\{i\in E:\, b_i>0,\ x^*(x_i)>0\}$, then we still have

$$\sum_{i\in E'} b_i x^*(x_i) > \frac{C}{2}\|y\|.$$

By homogeneity, we may also assume that $\min\{b_i : i \in E'\} = 1$, and hence $1\leq b_i\leq D$ for all $i\in E'$. Finally, let $k'\geq\min E'$ be minimal so that

$$\sum_{\substack{i\in E'\\ i>k'}} b_i x^*(x_i) \leq \gamma(k').$$

Then we have
$$\sum_{i\in E'} b_i x^*(x_i) \le (1+\epsilon) \sum_{\substack{i\in E'\\ i\ge k'}} b_i x^*(x_i).$$
Indeed, this is clear when $k'=\min E'$, whereas if $k'>\min E'$, then by the triangle-inequality, by (7) and by the choice of k' we have
$$\begin{aligned}\sum_{i\in E'} b_i x^*(x_i) &\le Dk'+\gamma(k') \le (1+\epsilon)\gamma(k'-1)\\ &\le (1+\epsilon)\sum_{\substack{i\in E'\\ i\ge k'}} b_i x^*(x_i),\end{aligned}$$
as claimed. Set $F=\{i\in E' : i\ge k'\}$. For each $i\in\mathbb{N}$ set $a_i=b_i$ when $i\ge\min F$ and $a_i=0$ when $i<\min F$, and let $\boldsymbol{a}=(a_i)_{i\in M}$. It is now routine to verify that $(\boldsymbol{a}, x^*, F)$ is a witness for M as defined above.

The next step is to select witnesses in a continuous manner using Lemma 2.5. Let Ω be the set of all witnesses of all infinite subsets of $\mathbb{N}$, and for each $M\in[\mathbb{N}]^\omega$ let $\Phi(M)$ be the (non-empty) set of all witnesses for M. For each $r\in\mathbb{N}$ let Ω_r be the set of elements $(\boldsymbol{a}, x^*, F)$ of Ω that satisfy $\max\operatorname{supp}(\boldsymbol{a})\le r$. It is easy to verify that the conditions of Lemma 2.5 are satisfied. It follows that there exists a function $\phi\colon[\mathbb{N}]^\omega\to\Omega$ such that $\phi(M)\in\Phi(M)$, i.e. $\phi(M)$ is a witness for M for all $M\in[\mathbb{N}]^\omega$, and ϕ is continuous if Ω is given the discrete topology. For each $M\in[\mathbb{N}]^\omega$ let $\phi(M)=(\boldsymbol{a}_M, x^*_M, F_M)$, and let n_M be the positive integer such that F_M has a Schreier decomposition $F_M=\bigcup_{j=1}^{n_M} F^M_j$ with $\operatorname{osc}(\boldsymbol{a}_M, F^M_j)\le d$ for each $j=1,\dots,n_M$. We will also use the notation
$$\boldsymbol{a}_M=(a^M_i),\quad x_M=\sum_{i\in M} a^M_i x_i,\quad\text{and}\quad A_M=\operatorname{supp}(\boldsymbol{a}_M).$$
By the proof of Lemma 2.5 we may assume that for each $M\in[\mathbb{N}]^\omega$ there is an $r\in\mathbb{N}$ such that $\Phi(M)\cap\Omega_s=\emptyset$ if $1\le s<r$ and $\phi(M)$ is the least element of $\Phi(M)\cap\Omega_r$ with respect to some fixed well-ordering of Ω. It follows that for $L,M\in[\mathbb{N}]^\omega$ if A_L is an initial segment of A_M, then we must have $\phi(L)=\phi(M)$. In particular
$$\mathcal{A}=\{A\in[\mathbb{N}]^{<\omega} : A=A_M \text{ for some } M\in[\mathbb{N}]^\omega\}$$
is a thin family, and we are in the situation of Corollary 2.4.

We shall now select infinite subsets $N_1\supset N_2\supset N_3$ of $\mathbb{N}$ stabilizing various parameters. To select N_1 we use Lemma 2.6. Let $f\colon[\mathbb{N}]^\omega\to c_0$ be the function mapping $M\in[\mathbb{N}]^\omega$ to $\big(x^*_M(x_i)\big)\in c_0$. Note that this is the only place, where we use the weakly null property of the sequence (x_i). It follows easily from the continuity of ϕ and from the w*-compactness of B_{X^*} that f is continuous with respect to the topology of pointwise convergence on c_0, and that the image of f has compact closure. Hence, by Lemma 2.6, there exists an infinite subset N_1 of $\mathbb{N}$ such that for all $P \in [N_1]^\omega$ we have

$$\sum_{i \in N_1 \setminus P} |x_P^*(x_i)| < \epsilon. \tag{14}$$

We next choose an infinite subset N_2 of N_1 using infinite Ramsey theory. We colour $\mathcal{A}$ by giving A_M, $M \in [\mathbb{N}]^\omega$, colour $(r_j)_{j=1}^{n_M} \in \mathbb{N}^{n_M}$ if

$$(1+\epsilon)^{r_j - 1} \leq \min\{a_i^M : i \in F_j^M\} < (1+\epsilon)^{r_j} \qquad \text{for each } j = 1, \dots, n_M.$$

This colouring is well-defined, i.e. the colour of $A \in \mathcal{A}$ does not depend on the choice of infinite set M with $A = A_M$. Note that for each $k \in \mathbb{N}$ the family $\{A \in \mathcal{A} : \min A = k\}$ is finitely coloured. An application of Proposition 2.3 now gives $N_2 \in [N_1]^\omega$ such that for all $L, M \in [N_2]^\omega$ if $\min F_L = \min F_M$, then $n_L = n_M$ and

$$\frac{a_i^M}{a_i^L} \geq \frac{1}{d(1+\epsilon)} \qquad \text{for all } i \in F_j^L \cap F_j^M,\ j = 1, \dots, n_L. \tag{15}$$

For our final stabilization we choose for each $k \in \mathbb{N}$ an ϵ/k-net S_k of $[0, \gamma(k) + D]^k$ (in the sense defined just before the start of this proof) together with an ordering of its elements. Given $M \in [\mathbb{N}]^\omega$, let $k = \min F_M$ and let $(w_j)_{j=1}^k$ be the least element of S_k satisfying

$$w_j \leq \sum_{i \in F_j^M} a_i^M x_M^*(x_i) \leq w_j + \epsilon/k \qquad \text{for each } j = 1, \dots, n_M.$$

We shall refer to $(w_j)_{j=1}^k$ as the *weight-colour* of M. This colouring of $[\mathbb{N}]^\omega$ induces a colouring of the family $\mathcal{A}$ satisfying the assumptions of Proposition 2.3. Hence there is an infinite subset N_3 of N_2 so that for all $L, M \in [N_3]^\omega$ if $\min F_L = \min F_M$ then L and M have the same weight-colour.

To finish the proof we apply the Schreier version of the Matching Lemma (Corollary 2.4). As observed earlier, the assumptions of the corollary are satisfied. So we can find $L, M \in [N_3]^\omega$ with $n_L = n_M$ such that

(i) for each $j = 1, \dots, n_L$ either $F_j^L \prec F_j^M$, or $F_j^M \prec F_j^L$, and

(ii) $L \cap M = \displaystyle\bigcup_{j=1}^{n_L} F_j^L \cap F_j^M$.

Note that in particular $\min F_L = \min F_M$, and hence L and M have the same weight-colour, say $(w_j)_{j=1}^k$, where $k=\min F_L$. Set

$$J = \big\{ j \in \{1,\dots,n_L\} : F_j^L \prec F_j^M \big\}.$$

Interchanging L and M and replacing J by $\{1,\dots,n_L\}\setminus J$ if necessary, we may assume that

$$\sum_{j\in J} w_j \geq \frac{1}{2}\sum_{j=1}^{n_L} w_j. \tag{16}$$

We now establish a number of inequalities. First we have

$$\begin{aligned}\|x_M\| \geq x_L^*(x_M) = \sum_{i\in M} a_i^M x_L^*(x_i) \geq \sum_{i\in L\cap M} a_i^M x_L^*(x_i) - \sum_{i\in M\setminus L} |a_i^M|\,|x_L^*(x_i)| \\ \geq \sum_{i\in L\cap M} a_i^M x_L^*(x_i) - 4\epsilon\|x_M\|,\end{aligned}$$

where the last inequality comes from (8) and from (14) applied with $P=L$. We now obtain the following sequence of inequalities (the steps are justified below).

$$\begin{aligned}\Big(1+4\epsilon\Big)\|x_M\| &\geq \sum_{i\in L\cap M} a_i^M x_L^*(x_i) \\ &\geq \frac{1}{d(1+\epsilon)}\sum_{j=1}^{n_L}\sum_{i\in F_j^L\cap F_j^M} a_i^L x_L^*(x_i) \\ &\geq \frac{1}{d(1+\epsilon)}\sum_{j\in J}\sum_{i\in F_j^L} a_i^L x_L^*(x_i) \\ &\geq \frac{1}{d(1+\epsilon)}\sum_{j\in J} w_j \geq \frac{1}{2d(1+\epsilon)}\sum_{j=1}^{n_L} w_j \\ &\geq \frac{1}{2d(1+\epsilon)}\Big(\sum_{j=1}^{n_L}\sum_{i\in F_j^M} a_i^M x_M^*(x_i) - \epsilon\Big) \\ &\geq \frac{1}{2d(1+\epsilon)}\frac{C}{2(1+\epsilon)(2+\epsilon)}\|x_M\| - \frac{\epsilon}{2d(1+\epsilon)}\,4\|x_M\|.\end{aligned}$$

The second line uses (15) and the third line uses the definition of J. For the next two lines we use the fact that L and M both have weight-colour $(w_j)_{j=1}^{n_L}$, and we also use (16). For the last inequality we apply (13) from the definition of a witness, and inequality (8) (from (12) we have $\|\boldsymbol{a}\|_{\ell_\infty} \geq 1$). We have thus shown that

$$C \leq 4d(1+\epsilon)^2(2+\epsilon)\Big(1+4\epsilon+\frac{2\epsilon}{d(1+\epsilon)}\Big).$$

Since ϵ was arbitrary it follows that $C \leq 8d$, as claimed. □

3. Schreier- and near-unconditionality

In this section we give new proofs of two results quoted in the Introduction. We begin with Schreier-unconditionality. It is not difficult to apply Theorem 2.1 with $d=1$ and a diagonal process to show that for any $\epsilon > 0$, every normalized, weakly null sequence has a Schreier-unconditional subsequence with constant $8+\epsilon$. The better constant claimed in Theorem 1.3 follows by a straightforward diagonal argument from the statement below. For $M \subset \mathbb{N}$ and $n \in \mathbb{N}$ we denote by $M^{(\leq n)}$ the collection of subsets of M of size at most n. So a sequence is $\mathbb{N}^{(\leq n)}$-unconditional if we can uniformly project onto sets of size at most n.

Theorem 3.1. *Fix $n \in \mathbb{N}$ and $\epsilon > 0$. Every normalized weakly null sequence has a $\mathbb{N}^{(\leq n)}$-unconditional subsequence with constant $1+\epsilon$.*

Proof. Let $C \in [1,\infty)$ and assume that (x_i) is a normalized weakly null sequence no subsequence of which is $\mathbb{N}^{(\leq n)}$-unconditional with constant C. We need to show that $C \leq 1$.

Let $M \in [\mathbb{N}]^\omega$. By our assumption there exists a triple $(\boldsymbol{a}, F, x^*)$, called a *witness for M*, such that

(17) $\boldsymbol{a} = (a_i) \in \mathrm{c}_{00}, \quad F \in M^{(\leq n)} \quad$ and $x^* \in B_{X^*}$;

(18) $\displaystyle\sum_{i \in M} a_i x_i \in S_X$;

(19) $\displaystyle\sum_{i \in F} a_i x^*(x_i) > C.$

Let Ω be the set of all witnesses of all infinite subsets of $\mathbb{N}$ equipped with the discrete topology. By Lemma 2.5 we obtain a continuous function $\phi\colon [\mathbb{N}]^\omega \to \Omega$ such that $\phi(M)$ is a witness for M for all $M \in [\mathbb{N}]^\omega$. For each $M \in [\mathbb{N}]^\omega$ we write

$$\phi(M) = (\boldsymbol{a}_M, F_M, x^*_M),$$

where $\boldsymbol{a}_M = (a_i^M)$, and we let $x_M = \sum_{i \in M} a_i^M x_i$.

We will now select infinite subsets $N_1 \supset N_2 \supset N_3$ of $\mathbb{N}$. We first choose N_1 so that $(x_i)_{i \in N_1}$ is a basic sequence with basis constant at most 2, say. Then in particular for any $M\in[N_1]^\omega$ we have

$$\sup_{i\in M}|a_i^M|\leq 4\Big\|\sum_{i\in M}a_i^M x_i\Big\|=4. \tag{20}$$

We next fix an arbitrary positive real number δ. We then select $N_2\in[N_1]^\omega$ so that for all $L,M\in[N_2]^\omega$ we have $|F_L|=|F_M|$ and

$$\big\|(a_i^L)_{i\in F_L}-(a_i^M)_{i\in F_M}\big\|_{\ell_1}<\delta. \tag{21}$$

This is done by a straightforward use of infinite Ramsey theory. Finally, using Lemma 2.6 we obtain $N_3\in[N_2]^\omega$ such that for all $P\in[N_3]^\omega$ we have

$$\sum_{i\in N_3\setminus P}|x_P^*(x_i)|<\delta. \tag{22}$$

After these stabilizations we apply Theorem 2.2 with $n=1$ to obtain infinite subsets L,M of N_3 such that either $F_L\prec F_M$ or $F_L\prec F_M$, and $L\cap M=F_L\cap F_M$. The choice of N_2 implies that in fact we have $F_L=F_M=L\cap M$. We now estimate $x_L^*(x_M)$ to obtain the required inequality. First, we write $x_L^*(x_M)$ as

$$\sum_{i\in M}a_i^M x_L^*(x_i)=\sum_{i\in F_L}a_i^L x_L^*(x_i)+\sum_{i\in F_M}\big(a_i^M-a_i^L\big)x_L^*(x_i) \\ +\sum_{i\in M\setminus F_M}a_i^M x_L^*(x_i). \tag{23}$$

We then estimate the three terms on the right-hand side of (23) as follows. Applying property (19) of a witness to L gives C as a lower bound on the first term. Applying (21) to the second term, and (20), (22) to the third term give upper bounds leading to

$$1=\|x_M\|\geq\big|x_L^*(x_M)\big|\geq C-\delta-4\delta.$$

Since δ was arbitrary, it follows that $C\leq 1$, as claimed. □

Remark. If (x_i) is a normalized basic sequence with basis constant C, then for all $(a_i)\in c_{00}$ we have

$$|a_n|=\Big\|\sum_{i=1}^n a_ix_i-\sum_{i=1}^{n-1}a_ix_i\Big\|\leq 2C\Big\|\sum_{i=1}^\infty a_ix_i\Big\|\qquad\text{for all } n\in\mathbb{N}.$$

We shall often use this to assume after passing to a subsequence (x_i) of a given normalized, weakly null sequence that $|a_n|\leq 4\big\|\sum_{i=1}^\infty a_ix_i\big\|$, say, for all $(a_i)\in c_{00}$ and for all $n\in\mathbb{N}$. The constant 4 is often adequate, however, sometimes we will need to be able to replace 4 by $1{+}\epsilon$ for any given $\epsilon>0$. We can do this by applying Theorem 3.1 with $n{=}1$.

We now turn to Elton's theorem on near-unconditional sequences. As mentioned in the Introduction, it is this result that raises Problem 1.2 — the main focus in this paper.

Proof of Theorem 1.1. Let $\delta\in(0,1]$ and let (x_i) be a normalized, weakly null sequence in a Banach space. An application of our main result, Theorem 2.1, with $d=D=1/\delta$ gives, for each $\epsilon>0$, a δ-near-unconditional subsequence of (x_i) with constant $8/\delta+\epsilon$. As we mentioned in the Introduction, a better constant of order $\log\big(1/\delta\big)$ can be obtained as follows. Set $d=D=2$ and pass to a (D,d)-bounded-oscillation-unconditional subsequence $(y_i)\subset(x_i)$ with constant 17, say. We show that (y_i) is δ-near-unconditional with constant $17k$, where $k=\big\lfloor\log_2\big(1/\delta\big)\big\rfloor+1$. Indeed, let $(a_i)\in c_{00}$ with $|a_i|\leq 1$ for all $i\in\mathbb{N}$, and let $E\subset\{i\in\mathbb{N}:\,|a_i|\geq\delta\}$. Set

$$E_j=\big\{i\in E:\,2^{-j}<|a_i|\leq 2^{-(j-1)}\big\},\qquad \text{for each } j=1,\dots,k.$$

Since $\operatorname{osc}((a_i),E_j)\leq 2$ we have

$$\Big\|\sum_{i\in E_j}a_iy_i\Big\|\leq 17\Big\|\sum_{i=1}^\infty a_iy_i\Big\|\qquad \text{for each } j=1,\dots,k.$$

Hence, by the triangle-inequality we get

$$\Big\|\sum_{i\in E}a_iy_i\Big\|\leq 17k\Big\|\sum_{i=1}^\infty a_iy_i\Big\|,$$

as claimed. Note that $17k<18\log_2\big(1/\delta\big)$ if δ is sufficiently small. □

Let us mention that recently Lopez-Abad and Todorcevic [17] also gave new proofs of Theorems 1.1 and 1.3 based on results on pre-compact families of finite subsets of $\mathbb{N}$.

We next show that a positive answer to Problem 1.2 implies a positive answer to Problem 1.5.

Proposition 3.2. *If $\sup_{\delta>0}K(\delta)<\infty$, then there exists a constant C such that every normalized, weakly null sequence has a quasi-greedy subsequence with constant C.*

Proof. Let $C>2\sup_{\delta>0}K(\delta)+1$. Fix $\epsilon\in(0,1)$, and for each $n\in\mathbb{N}$ set $\delta_n=\epsilon/n$. Given a normalized, weakly null sequence (x_i), we apply a diagonal procedure to extract a subsequence (y_i) such that for each $n\in\mathbb{N}$ the tail $(y_i)_{i=n}^\infty$ is δ_n-near-unconditional with constant $K(\delta_n)+\epsilon$. Passing to a further subsequence, if necessary, we may assume that (y_i) is a basic sequence with constant $1+\epsilon$, and moreover $|a_n|\leq(1+\epsilon)\big\|\sum_{i=1}^\infty a_iy_i\big\|$ for all $(a_i)\in c_{00}$ and for all $n\in\mathbb{N}$. For the latter property we used Theorem 3.1 with $n=1$. We will now show that (y_i) is quasi-greedy with constant C provided ϵ is sufficiently small.

Given $(a_i)\in c_{00}$ and $\delta\in(0,1]$, we need to show that

$$\Big\|\sum_{i\in E} a_i y_i\Big\| \le C\|x\|, \tag{24}$$

where $E=\{i\in\mathbb{N}:\ |a_i|\ge\delta\}$ and $x=\sum_{i=1}^\infty a_i y_i$. We may clearly assume that $\sup_i|a_i|=1$, which implies that $\|x\|\ge(1+\epsilon)^{-1}>1/2$. Now choose the smallest $n\in\mathbb{N}$ such that $\delta_n\le\delta$. Note that $(n-1)\delta\le\epsilon<2\epsilon\|x\|$. Hence

$$\begin{aligned}\Big\|\sum_{i\in E,\ i<n} a_i y_i\Big\| &\le \Big\|\sum_{i=1}^{n-1} a_i y_i\Big\| + \Big\|\sum_{i\notin E,\ i<n} a_i y_i\Big\| \\ &\le (1+\epsilon)\|x\| + (n-1)\delta \le (1+3\epsilon)\|x\|.\end{aligned}$$

On the other hand, since $(y_i)_{i=n}^\infty$ is δ_n-near-unconditional with constant $K(\delta_n)+\epsilon$, we have

$$\Big\|\sum_{i\in E,\ i\ge n} a_i y_i\Big\| \le \big(K(\delta_n)+\epsilon\big)\Big\|\sum_{i=n}^\infty a_i y_i\Big\| \le \big(K(\delta_n)+\epsilon\big)(2+\epsilon)\|x\|.$$

Now (24) follows for suitable ϵ by the triangle inequality. □

We conclude this section by presenting a positive answer to Problem 1.2 when we are "far" from c_0. As a corollary we obtain a positive answer to Problem 1.5 in this case which was shown in [10]. The argument here follows the proof in [10].

Theorem 3.3. *If (x_i) is a normalized, weakly null sequence with spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon>0$ and for any $\delta\in(0,1)$ there is a δ-near-unconditional subsequence of (x_i) with constant $1+\epsilon$.*

Proof. Fix $\delta\in(0,1)$. We will show that after passing to a subsequence of (x_i) there exists $N\in\mathbb{N}$ such that for all $(a_i)\in c_{00}$ if $\|\sum a_i x_i\|\le 1$, then the set $\{i\in\mathbb{N}:\ |a_i|\ge\delta\}$ has size at most N. We can then finish the proof as follows. We first apply Theorem 3.1 to obtain a $\mathbb{N}^{(\le N)}$-unconditional subsequence (y_i) of (x_i) with constant $1+\epsilon$. This subsequence has the property that for all $(a_i)\in c_{00}$ with $\|\sum a_i y_i\|\le 1$ and for all $E\subset\{i\in\mathbb{N}:\ |a_i|\ge\delta\}$

$$\Big\|\sum_{i\in E} a_i y_i\Big\| \le (1+\epsilon)\Big\|\sum_{i=1}^\infty a_i y_i\Big\|\ .$$

Since we can do this for every $\delta\in(0,1)$ and for any $\epsilon>0$, we can use the proof of Proposition 4.2 below (see also Definition 4.1) to pass to a further subsequence that is δ-near-unconditional with constant $1+\epsilon$.

It remains to show the initial claim. Let (e_i) be the spreading model of (x_i). After passing to a subsequence of (x_i), we can assume that (x_i) is a basic sequence with constant 2, and for positive integers $n\leq k_1<\ldots<k_n$ and for real numbers $(a_i)_{i=1}^n$ we have

$$\Big|\big\|\sum_{i=1}^n a_i x_{k_i}\big\| - \big\|\sum_{i=1}^n a_i e_i\big\|\Big| < \big\|\sum_{i=1}^n a_i x_{k_i}\big\| \ .$$

Assume that for all $n\in\mathbb{N}$ there exists $(a_i^n)_i\in c_{00}$ such that $\|\sum_i a_i^n x_i\|\leq 1$ and the set $\{i\in\mathbb{N} : |a_i^n|\geq\delta\}$ has size at least $2n$. It follows that $\|\sum_{i\geq n} a_i^n x_i\|\leq 3$ and the set $\{i\geq n : |a_i^n|\geq\delta\}$ has size at least n. Since (e_i) is 1-spreading and 1-surpression-unconditional, we deduce that $\|\sum_{i=1}^n e_i\|\leq 12/\delta$ for all $n\in\mathbb{N}$. From here, it is easy to show that (e_i) is $24/\delta$-equivalent to the unit vector basis of c_0. This contradiction completes the proof. □

Combining he above theorem with the proof of Proposition 3.2 we obtain

Corollary 3.4 (Dilworth, Kalton and Kutzarova [10]). *If (x_i) is a normalized, weakly null sequence with spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon>0$ there is a quasi-greedy subsequence of (x_i) with constant $3+\epsilon$.*

4. Variants of near-unconditionality

In the following sections we will be considering various problems that turn out to be related to the Elton problem. In order to make this relationship precise we will now introduce some variants of the constant $K(\delta)$, and explain the relationships between them. To begin with we recall the definition of $K(\delta)$ in a slightly different way. Given $\delta\in(0,1]$ and a normalized, weakly null sequence (x_i), let $K((x_i),\delta)$ be the least real number C such that (x_i) is δ-near-unconditional with constant C,i.e. for all $(a_i)\in c_{00}$ and for all $E\subset\{i\in\mathbb{N} : |a_i|\geq\delta\}$ if $\sup_i|a_i|\leq 1$, then $\big\|\sum_{i\in E} a_i x_i\big\|\leq C\big\|\sum_{i=1}^\infty a_i x_i\big\|$. Observe that

$$K(\delta)=\sup_{(x_i)}\inf_{(y_i)\subset(x_i)} K((y_i),\delta),$$

where the supremum is taken over all normalized, weakly null sequences (x_i) and the infimum over all subsequences (y_i) of (x_i). Recall that the normalization $\sup_i|a_i|\leq 1$ in the definition is essential (see remarks in the Introduction). We will now introduce three other constants K', L and L'. For K' we will use the normalization $\|\sum_i a_i x_i\| \le 1$, whereas in the definition of L, L' we restrict to vectors all whose non-zero coefficients are "large". Below we repeated the definition of K for the convenience of the reader.

Definition 4.1. *Let $\delta \in (0,1]$ and let (x_i) be a normalized, weakly null sequence in a Banach space. Each supremum below is over all normalized, weakly null sequences (y_i) and the infimum is taken over all subsequences (z_i) of (y_i).*

$$K((x_i),\delta) = \inf\Big\{C : \Big\|\sum_{i\in E} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \text{ whenever } (a_i)\in c_{00},\\ E\subset\{i\in\mathbb{N} : |a_i|\ge\delta\},\ \sup_i |a_i|\le 1\Big\}$$

$$K(\delta) = \sup_{(y_i)} \inf_{(z_i)\subset(y_i)} K((z_i),\delta)$$

$$K'((x_i),\delta) = \inf\Big\{C : \Big\|\sum_{i\in E} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \text{ whenever } (a_i)\in c_{00},\\ E\subset\{i\in\mathbb{N} : |a_i|\ge\delta\},\ \Big\|\sum_{i=1}^{\infty} a_i x_i\Big\|\le 1\Big\}$$

$$K'(\delta) = \sup_{(y_i)} \inf_{(z_i)\subset(y_i)} K'((z_i),\delta)$$

$$L((x_i),\delta) = \inf\Big\{C : \Big\|\sum_{i\in E} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \text{ whenever } \boldsymbol{a}=(a_i)\in c_{00},\\ |a_i|\ge\delta\ \forall i\in\operatorname{supp}(\boldsymbol{a}),\ E\in[\mathbb{N}]^{<\omega},\ \sup_i |a_i|\le 1\Big\}$$

$$L(\delta) = \sup_{(y_i)} \inf_{(z_i)\subset(y_i)} L((z_i),\delta)$$

$$L'((x_i),\delta) = \inf\Big\{C:\ \Big\|\sum_{i\in E} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^\infty a_i x_i\Big\| \text{ whenever } \boldsymbol{a}=(a_i)\in c_{00},$$
$$|a_i|\ge\delta\ \forall i\in\operatorname{supp}(\boldsymbol{a}),\ E\in[\mathbb{N}]^{<\omega},\ \Big\|\sum_{i=1}^\infty a_i x_i\Big\|\le 1\Big\}$$

$$L'(\delta) = \sup_{(y_i)} \inf_{(z_i)\subset(y_i)} L'((z_i),\delta)$$

The following result establishes some relationships between the constants we just introduced. It shows in particular that for solving Problem 1.2 we are free to choose the normalization. In many situations it is more convenient to work with the constants K' and L' instead of K and L.

Proposition 4.2. *Let K, K', L and L' be the functions defined above.*

(i) If $0<\delta_1<\delta_2\le 1$, then $K'(\delta_2)\le K(\delta_1)$ and $L'(\delta_2)\le L(\delta_1)$.
(ii) If $0<\delta\le 1$, then $L(\delta)\le K(\delta)$ and $L'(\delta)\le K'(\delta)$.

In particular we have $\sup_{\delta>0} K(\delta)=\sup_{\delta>0} K'(\delta)\ge\sup_{\delta>0} L(\delta)=\sup_{\delta>0} L'(\delta)$.

Proof. (ii) is clear from definition. To see (i) let (x_i) be a normalized, weakly null sequence. By Theorem 3.1 we may assume, after passing to a subsequence if necessary, that

$$\sup_i |a_i| \le \frac{\delta_2}{\delta_1}\Big\|\sum_{i=1}^\infty a_i x_i\Big\| \qquad \text{for all } (a_i)\in c_{00}.$$

Now given $(a_i)\in c_{00}$ with $\big\|\sum_i a_i x_i\big\|\le 1$ and $E\subset\{i\in\mathbb{N}:\ |a_i|\ge\delta_2\}$, if $b_i=\frac{\delta_1}{\delta_2}a_i$ for all $i\in\mathbb{N}$, then $\sup_i|b_i|\le 1$ and $E\subset\{i\in\mathbb{N}:\ |b_i|\ge\delta_1\}$. It follows that for any subsequence (y_i) of (x_i) we have $K'((y_i),\delta_2)\le K((y_i),\delta_1)$ and $L'((y_i),\delta_2)\le L((y_i),\delta_1)$.

It remains to show that $\sup_\delta K(\delta)\le\sup_\delta K'(\delta)$ and $\sup_\delta L(\delta)\le\sup_\delta L'(\delta)$. We show the second inequality (the proof of the first one is similar). Assume that $L'=\sup_\delta L'(\delta)<\infty$, and let $\delta\in(0,1]$. We will show that $L(\delta)\le L'$. Let (x_i) be a normalized, weakly null sequence. Fix $\epsilon\in(0,1]$ and positive real numbers M_n such that $n<\epsilon(M_n-1)$ for all $n\in\mathbb{N}$. After passing to a subsequence, if necessary, we may assume that (x_i) is a basic sequence with constant $1+\epsilon$. Then using a standard diagonal argument we pass to a subsequence (y_i) of (x_i) such that for each $n\in\mathbb{N}$ we have $L'\big((y_i)_{i\ge n},\delta/M_n\big)\le L'+\epsilon$. We claim that $L((y_i),\delta)\le(L'+2\epsilon)(1+3\epsilon)$.

Given $\boldsymbol{a}=(a_i)\in \mathrm{c}_{00}$ with $\delta\le |a_i|\le 1$ for all $i\in\mathrm{supp}(\boldsymbol{a})$, and $E\in[\mathbb{N}]^{<\omega}$, set $x=\sum_i a_iy_i$ and choose $n\in\mathbb{N}$ minimal so that

$$\Big\|\sum_{i\ge n} a_iy_i\Big\|\le M_n.$$

Note that $\big\|\sum_{i\ge n} a_iy_i\big\|\ge (n-1)/\epsilon$. Now by the choice of (y_i) we have

$$\Big\|\sum_{\substack{i\in E\\ i\ge n}} a_iy_i\Big\|\le (L'+\epsilon)\Big\|\sum_{i\ge n} a_iy_i\Big\|,$$

and hence

$$\begin{aligned}\Big\|\sum_{i\in E} a_iy_i\Big\| &\le (n-1)+(L'+\epsilon)\Big\|\sum_{i\ge n} a_iy_i\Big\|\\ &\le (L'+2\epsilon)\Big\|\sum_{i\ge n} a_iy_i\Big\|\\ &\le (L'+2\epsilon)\big(\|x\|+n-1\big)\\ &\le (L'+2\epsilon)(1+3\epsilon)\|x\|.\end{aligned}$$

Indeed, $n-1\le\epsilon\big\|\sum_{i\ge n} a_iy_i\big\|\le\epsilon(2+\epsilon)\|x\|\le 3\epsilon\|x\|$ since (y_i) is a basic sequence with constant $1+\epsilon$. We have thus proved our claim from which $L(\delta)\le L'$ follows. $\square$

To conclude this section we show that the various constants we introduced remain the same if we restrict to the class of Banach spaces $C(S)$, where S is a countable, compact metric space. Recall that such a space S is homeomorphic to a countable successor ordinal in its order topology.

Theorem 4.3. *For each $\delta\in(0,1]$, we have*

$$K(\delta)=\sup_{\alpha,(x_i)}\ \inf_{(y_i)\subset(x_i)} K((y_i),\delta),$$

where the supremum is taken over all countable, successor ordinals α and all normalized, weakly null sequences (x_i) in $C(\alpha)$, and the infimum is taken over all subsequences (y_i) of (x_i). The analogous statements for the functions K', L and L' also hold.

Proof. We prove the result only for K. The argument for the other functions is similar. Fix $\epsilon\in(0,1]$. By the definition of $K(\delta)$ there is a normalized, weakly null sequence (x_i) in some Banach space such that $K((y_i),\delta)>K(\delta)-\epsilon$ for every subsequence (y_i) of (x_i). Thus for each $M\in[\mathbb{N}]^{\omega}$ we have a triple $(\boldsymbol{a},E,x^*)$ witnessing $K((x_i)_{i\in M},\delta)>K(\delta)-\epsilon$, that is

$$\begin{aligned}
&(25)\quad \boldsymbol{a}=(a_i)\in \mathrm{c}_{00},\quad E\subset\{i\in M:\ |a_i|\geq\delta\},\quad x^*\in B_{X^*},\\
&(26)\quad \sup_{i\in M}|a_i|=1,\\
&(27)\quad \sum_{i\in E}a_i x^*(x_i)>(K(\delta)-\epsilon)\|x\|,\quad \text{where } x=\sum_{i\in M}a_i x_i.
\end{aligned}$$

We now use Lemma 2.5 to obtain a continuous selection $M\mapsto(\boldsymbol{a}_M,E_M,x^*_M)$ of witnesses in the usual way. We set $\boldsymbol{a}_M=(a_i^M)$ and $x_M=\sum_{i\in M}a_i^M x_i$.

Next we pass to infinite subsets $N_1\supset N_2\supset N_3$ of $\mathbb{N}$. First, there exists $N_1\in[\mathbb{N}]^\omega$ such that $(x_i)_{i\in N_1}$ is a basic sequence. Then we use Theorem 3.1 with $n=1$ to find $N_2\in[N_1]^\omega$ such that $\sup_{i\in N_2}|a_i|\leq(1+\epsilon)\big\|\sum_{i\in N_2}a_i x_i\big\|$ for all $(a_i)\in\mathrm{c}_{00}$. Note that in particular we have $\|x_M\|\geq(1+\epsilon)^{-1}$ for all $M\in[N_2]^\omega$. Finally, by Lemma 2.6 there exists $N_3\in[N_2]^\omega$ such that $\sum_{i\in N_3\setminus P}|x^*_P(x_i)|<\epsilon$ for all $P\in[N_3]^\omega$.

After relabelling, if necessary, we can take $N_3=\mathbb{N}$. Set $X=[x_i]_{i=1}^\infty$, and let (x_i^*) be the biorthogonal functionals to (x_i). Note that $\|x_i^*\|\leq 1+\epsilon$ for all $i\in\mathbb{N}$ by the choice of N_2. For each $i\in\mathbb{N}$ and for $t\in[-1,1]$ define $\rho_i(t)=\frac{\epsilon}{2^i}\big\lfloor\frac{2^i t}{\epsilon}\big\rfloor\vee(-1)$, and note that $|\rho_i(t)-t|\leq\epsilon/2^i$. Now for each $M\in[\mathbb{N}]^\omega$ we have $\sum_{i\in\mathbb{N}\setminus M}|x^*_M(x_i)|<\epsilon$ by the choice of N_3, and $x^*_M=\sum_{i=1}^\infty x^*_M(x_i)x_i^*$ in the weak-$*$-sense since (x_i) is a basis for X. It follows that

$$\widetilde{x^*_M}=\sum_{i\in M}\rho_i(x^*_M(x_i))x_i^*,$$

converges in the weak-$*$ sense, and moreover

$$\big\|\widetilde{x^*_M}\big\|\leq 1+2\epsilon(1+\epsilon)\leq 1+4\epsilon,\qquad \text{for all } M\in[\mathbb{N}]^\omega.$$

Now define S to be the closure of $U=\big\{\widetilde{x^*_M}:\ M\in[\mathbb{N}]^\omega\big\}\cup\{x_i^*:\ i\in\mathbb{N}\}$ in the weak-$*$ topology. Since U is bounded in norm, S is a compact metric space. The continuity of the choice of witnesses implies that U is countable, and hence, because of the discretization of coefficients using the functions ρ_i, S is also countable.

Let $T\colon X\to C(S)$ be the canonical map, i.e. $T(x)(y^*)=y^*(x)$ for all $x\in X$, $y^*\in S$, and note that $\|T\|\leq 1+4\epsilon$. Set $f_i=T(x_i)$ for all $i\in\mathbb{N}$, and $f_M=T(x_M)$ for all $M\in[\mathbb{N}]^\omega$. Then (f_i) is a normalized, weakly null sequence in $C(S)$. We claim that $K((f_i)_{i\in M},\delta)\geq\big(K(\delta)-3\epsilon\big)(1+4\epsilon)^{-1}$ for all $M\in[\mathbb{N}]^\omega$, which proves the assertion of the theorem.

For $M\in[\mathbb{N}]^\omega$ we have $f_M=\sum_{i\in M}a_i^M f_i$ and $\|f_M\|\leq(1+4\epsilon)\|x_M\|$. We also have $E_M\subset\{i\in M:\,|a_i^M|\geq\delta\}$, and

$$\begin{aligned}\Big\|\sum_{i\in E_M}a_i^M f_i\Big\| &\geq \sum_{i\in E_M}a_i^M f_i\big(\widetilde{x_M^*}\big)=\sum_{i\in E_M}a_i^M\rho_i(x_M^*(x_i))\\ &\geq \sum_{i\in E_M}a_i^M x_M^*(x_i)-\epsilon\\ &> (K(\delta)-\epsilon)\|x_M\|-\epsilon\geq(K(\delta)-3\epsilon)\|x_M\|\\ &\geq \big(K(\delta)-3\epsilon\big)(1+4\epsilon)^{-1}\|f_M\|,\end{aligned}$$

as required. □

5. The c_0-problem

In this short section we consider the following intriguing question which, to our knowledge, has not been raised elsewhere.

Problem 5.1. *Is there a real number C such that every sequence equivalent to the unit vector basis of c_0 has an unconditional subsequence with constant C?*

Let Y be the space c_0 or ℓ_p for some $p\in[1,\infty)$, and let (e_i) be the unit vector basis of Y. Let (x_i) be a sequence in a Banach space equivalent to (e_i). A well known result of James [15] says that if $Y=c_0$ or $Y=\ell_1$, then for any $\epsilon>0$ there is a block basis of (x_i) that is $(1+\epsilon)$-equivalent to (e_i), and so in particular there is a block basis of (x_i) that is unconditional with constant $(1+\epsilon)$. Both these conclusions fail spectacularly if $Y=\ell_p$ for some $p\in(1,\infty)$: for any constant C there is an equivalent norm on Y so that it contains no unconditional basic sequence with constant C. This follows from the solution of the distortion problem by Odell and Schlumprecht [22]. For c_0 and ℓ_1 one can go further and consider subsequences instead of block bases. However, if $Y=c_0$, then for any C there are easy examples that show that (x_i) does not need to have a subsequence C-equivalent to (e_i). If $Y=\ell_1$, then for any constant C there are easy examples that show that (x_i) does not even need to have an unconditional subsequence with constant C. The only remaining question in this context is raised in Problem 5.1, which is still open. Example 8.7 in Section 8 will show (among other things) that Problem 5.1 cannot have a positive answer with $C<5/4$. However, it is possible that a uniform constant C exists. Indeed, this happens if and only if $\sup_{\delta>0}L'(\delta)<\infty$, where L' is the function given in Definition 4.1. Our aim in this section is to prove this equivalence.

For each $\delta \in (0,1]$ let us define $C(\delta)$ to be the infimum of the set of real numbers C such that every normalized sequence $1/\delta$-equivalent to the unit vector basis of c_0 has an unconditional subsequence with constant C. So a positive answer to Problem 5.1 is equivalent to the statement that $\sup_{\delta>0} C(\delta)$ is finite.

Theorem 5.2. *Let $\delta, \delta_1 \in (0,1]$.*

(i) If $\delta_1 \le \delta$, then $C(\delta) \le L(\delta_1) \cdot \Big(1 + \dfrac{\delta_1}{\delta}\Big) + \dfrac{\delta_1}{\delta}$.

(ii) If $\delta_1 < \dfrac{\delta}{2L'(\delta)}$, then $L'(\delta) \le C(\delta_1)$.

In particular we have $\sup_{\delta>0} C(\delta) = \sup_{\delta>0} L'(\delta) = \sup_{\delta>0} L(\delta)$.

Proof. To verify (i) fix $\epsilon \in (0,1]$, and assume that (x_i) is a normalized sequence $1/\delta$-equivalent to the unit vector basis of c_0. So for some constants $A>0$ and $B>0$ with $B/A \le 1/\delta$ we have

$$A \sup_i |a_i| \le \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\| \le B \sup_i |a_i| \tag{28}$$

for all $(a_i) \in \mathrm{c}_{00}$. After passing to a subsequence, if necessary, we may assume that $L((x_i), \delta_1) \le L(\delta_1) + \epsilon$. We claim that under these circumstances (x_i) is unconditional with constant $C = \big(L(\delta_1) + \epsilon\big)\Big(1 + \frac{\delta_1}{\delta}\Big) + \frac{\delta_1}{\delta}$, from which (i) follows.

Given $(a_i) \in \mathrm{c}_{00}$ and $A \in [\mathbb{N}]^{<\omega}$, we need to show that $\big\| \sum_{i \in A} a_i x_i \big\| \le C\|x\|$, where $x = \sum_i a_i x_i$. We may clearly assume that $\sup_i |a_i| = 1$. Then it follows from (28) that

$$B\delta \le A = A \sup_i |a_i| \le \|x\|,$$

and hence for every $F \subset \mathbb{N}$ we have

$$\Big\| \sum_{i \in F} a_i x_i \Big\| \le B \sup_{i \in F} |a_i| \le \frac{\|x\|}{\delta} \sup_{i \in F} |a_i|. \tag{29}$$

Set $E = \{i \in \mathbb{N} : |a_i| \ge \delta_1\}$. The definition of $L((x_i), \delta_1)$ gives

$$\Big\| \sum_{i \in A \cap E} a_i x_i \Big\| \le \big(L(\delta_1) + \epsilon\big) \Big\| \sum_{i \in E} a_i x_i \Big\|.$$

Then from (29) we get

$$\begin{aligned}\Big\|\sum_{i\in E} a_ix_i\Big\| &\le \|x\| + \Big\|\sum_{i\notin E} a_ix_i\Big\| \le \Big(1+\frac{\delta_1}{\delta}\Big)\|x\|,\\ \Big\|\sum_{i\in A\setminus E} a_ix_i\Big\| &\le \frac{\delta_1}{\delta}\,\|x\|.\end{aligned}$$

Finally, by the triangle-inequality we obtain

$$\Big\|\sum_{i\in A} a_ix_i\Big\| \le \Big[\big(L(\delta_1)+\epsilon\big)\Big(1+\frac{\delta_1}{\delta}\Big)+\frac{\delta_1}{\delta}\Big]\cdot\|x\|$$

as required.

We now prove (ii). Fix $\epsilon>0$. By the definition of $L'(\delta)$ there is a normalized, weakly null sequence (x_i) such that $L'((y_i),\delta)>L'(\delta)-\epsilon$ for all subsequences (y_i) of (x_i). So for each $M\in[\mathbb{N}]^\omega$ there is a triple $(\boldsymbol{a},E,x^*)$ that we shall call a *witness for* M, where

(30) $\boldsymbol{a}=(a_i)\in c_{00},\quad E\subset\operatorname{supp}(\boldsymbol{a})\subset M,\quad x^*\in B_{X^*};$

(31) $|a_i|\ge\delta$ for all $i\in\operatorname{supp}(\boldsymbol{a})$, and $\|x\|\le 1$, where $x=\sum_{i\in M} a_ix_i$;

(32) $\sum_{i\in E} a_ix^*(x_i) > L'(\delta)-\epsilon.$

Let Ω be the set of all witnesses of all infinite subsets of $\mathbb{N}$, and for $M\in[\mathbb{N}]^\omega$ let $\Phi(M)$ be the (nonempty) set of all witnesses for M. For $r\in\mathbb{N}$ let Ω_r be the set of all triples $(\boldsymbol{a},E,x^*)\in\Omega$ such that $\max\operatorname{supp}(\boldsymbol{a})\le r$. By Lemma 2.5 there is a function $\phi\colon[\mathbb{N}]^\omega\to\Omega$ such that $\phi(M)$ is a witness for M for all $M\in[\mathbb{N}]^\omega$, and ϕ is continuous when Ω is given the discrete topology. We let $\phi(M)=(\boldsymbol{a}_M,E_M,x^*_M)$ and let

$$A_M=\operatorname{supp}(\boldsymbol{a}_M),\quad \boldsymbol{a}_M=(a_i^M),\quad x_M=\sum_{i\in M} a_i^Mx_i.$$

By the proof of Lemma 2.5 we can choose ϕ so that for all $M\in[\mathbb{N}]^\omega$ there exists $r\in\mathbb{N}$ such that $\Phi(M)\cap\Omega_s=\emptyset$ whenever $1\le s<r$, and $\phi(M)$ is the least element of $\Phi(M)\cap\Omega_r$ in some well-ordering of Ω fixed in advance. It is then easy to verify that for all $L,M\in[\mathbb{N}]^\omega$ if $A_M\prec L$, then $\phi(L)=\phi(M)$.

We now pass to infinite subsets $N_1\supset N_2\supset N_3\supset N_4$ of $\mathbb{N}$. Let $f\colon[\mathbb{N}]^\omega\to c_0$ be the function that maps $M\in[\mathbb{N}]^\omega$ to $f_M=\big(x^*_M(x_i)\big)\in c_0$. It follows from the continuity of ϕ that f is continuous with respect to the topology of pointwise convergence on c_0 and that its image has compact closure. Hence by Lemma 2.6

there exists $N_1 \in [\mathbb{N}]^\omega$ such that

$$\sum_{i\in N_1\setminus P} |x_P^*(x_i)| < \epsilon \qquad \text{for all } P\in [N_1]^\omega. \tag{33}$$

Now choose arbitrary $N_2 \in [N_1]^\omega$ with $N_1 \setminus N_2$ of infinite size. Given $M\in[N_2]^\omega$, we can choose $L\in[N_1]^\omega$ such that $A_M \prec L$ and $L\setminus A_M \subset N_1\setminus N_2$. Then $\phi(L)=\phi(M)$, and applying (33) with $P=L$ we obtain

$$\sum_{i\in N_2\setminus A_M} |x_M^*(x_i)| = \sum_{i\in N_2\setminus A_M} |x_L^*(x_i)| < \epsilon \tag{34}$$

since $N_2\setminus A_M \subset N_1 \setminus L$. In other words, relative to N_2 and up to a small error, we have $\operatorname{supp}(x_M^*) \subset A_M$ for all $M\in[N_2]^\omega$.

By the definition of $L'(\delta)$ there exists $N_3\in[N_2]^\omega$ such that $L'((x_i)_{i\in N_3},\delta) \leq L'(\delta)+\epsilon$. Finally, we apply Theorem 3.1 with $n=1$ to obtain $N_4\in[N_3]^\omega$ such that for all $M\in[N_4]^\omega$ we have $|a_i^M| \leq 1+\epsilon$ for all $i\in M$.

We now relabel so that we can take $N_4=\mathbb{N}$, and define a new norm on c_0 by setting

$$|||\boldsymbol{b}||| = \|\boldsymbol{b}\|_{\mathrm{c}_0} \vee \sup_{M\in[\mathbb{N}]^\omega} \Big| \sum_{i=1}^\infty b_i x_M^*(x_i) \Big| \qquad \text{for } \boldsymbol{b}=(b_i)\in \mathrm{c}_0.$$

Let (y_i) be the unit vector basis of c_0 considered with its new norm. It follows from (34) and the choice of N_3 that

$$\delta \|x_M^*\|_{\ell_1} \leq \sum_{i\in A_M} \big|a_i^M\big| \big|x_M^*(x_i)\big| + \epsilon\delta \leq 2\big(L'(\delta)+\epsilon\big) + \epsilon\delta$$

for all $M\in[\mathbb{N}]^\omega$. Hence (y_i) is D-equivalent to (e_i), where

$$D = \frac{2\big(L'(\delta)+\epsilon\big)+\epsilon\delta}{\delta} < \frac{1}{\delta_1}$$

provided ϵ is sufficiently small. We claim that (y_i) has no unconditional subsequence with constant $C=\big(L'(\delta)-\epsilon\big)/(1+\epsilon)$. Fix $M\in[\mathbb{N}]^\omega$. We have

$$\Big|\sum_{i\in M} a_i^M x_L^*(x_i)\Big| = |x_L^*(x_M)| \leq 1 \qquad \text{for all } L\in[\mathbb{N}]^\omega,$$

and hence, by the choice of N_4, we have $|||\boldsymbol{a}_M||| \leq 1+\epsilon$. On the other hand, property (32) of a witness applied to M gives

$$\Big|\Big|\Big| \sum_{i\in E_M} a_i^M y_i \Big|\Big|\Big| \geq \sum_{i\in E_M} a_i^M x_M^*(x_i) > L'(\delta)-\epsilon \geq C|||\boldsymbol{a}_M|||,$$

which shows the claim. Since ϵ was arbitrary, $C(\delta_1)\geq L'(\delta)$ follows.

Parts (i) and (ii) show that $\sup_{\delta>0} L'(\delta) \leq \sup_{\delta>0} C(\delta) \leq \sup_{\delta>0} L(\delta)$. That we have equality throughout follows from Proposition 4.2. $\square$

6. Convex-unconditionality and duality

The following notion of partial unconditionality was introduced by Argyros, Mercourakis, and Tsarpalias [5]. Given $\delta \in (0,1]$, we say that a basic sequence (x_i) is *δ-convex-unconditional with constant A* if for all $(a_i) \in c_{00}$ and for all $E \in [\mathbb{N}]^{<\omega}$ if

$$\delta \sum_{i\in E} |a_i| \leq \Big\| \sum_{i\in E} a_i x_i \Big\|,$$

then we have

$$\Big\| \sum_{i\in E} a_i x_i \Big\| \leq A \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\|.$$

The definition in [5] is actually slightly different, but it is equivalent to ours (they express unconditionality in terms of sign-changes rather than projections). Theorem 1.4 on ℓ_1-projections follows immediately from the next result.

Theorem 6.1 (Argyros, Mercourakis, and Tsarpalias [5]). *Given $\delta \in (0,1]$ there is a constant A such that every normalized weakly null sequence has a δ-convex-unconditional subsequence with constant A. Moreover, $A \leq 16 \log_2 \big(1/\delta\big)$ for $\delta < 1/4$.*

Proof. Given $\delta \in (0,1]$, define $l = \big\lfloor \log_2 \big(1/\delta\big) \big\rfloor + 2$ and fix $A \in [1,\infty)$. Assume that (x_i) is a normalized, weakly null sequence, which has no δ-convex-unconditional subsequence with constant A. We will show that $A \leq 8l$.

Without loss of generality (x_i) is a basic sequence with constant 2, say. So for all $(a_i) \in c_{00}$ we have

$$\sup_i |a_i| \leq 4 \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\|. \tag{35}$$

Let $M \in [\mathbb{N}]^{\omega}$. Since $(x_i)_{i\in M}$ is not δ-convex-unconditional with constant A, there exist $(a_i) \in c_{00}$ and $E \in [M]^{<\omega}$ such that

$$\delta \sum_{i\in E} |a_i| \leq \Big\| \sum_{i\in E} a_i x_i \Big\|, \quad \text{and} \quad A\|x\| < \Big\| \sum_{i\in E} a_i x_i \Big\|,$$

where $x = \sum_{i\in M} a_i x_i$. Rescaling, considering appropriate subsets of E, and replacing (a_i) by $(-a_i)$ if necessary, we conclude that for every $M \in [\mathbb{N}]^{\omega}$ there exists a quadruple $(\boldsymbol{a}, F, x^*, k)$, called a *witness for M*, with the following properties.

(36) $\boldsymbol{a}=(a_i)\in c_{00}, \quad F\in[M]^{<\omega}, \quad x^*\in B_{X^*}, \quad k\in\{1,\dots,l\}$,

(37) $a_i>0$ and $2^{-k}<x^*(x_i)\leq 2^{-k+1}$ for all $i\in F$,

(38) $\displaystyle\frac{\delta}{4l}\leq\sum_{i\in F}a_i x^*(x_i)$,

(39) $\displaystyle A\|x\|<2l\sum_{i\in F}a_i x^*(x_i)+\frac{\delta}{2}, \qquad \text{where } x=\sum_{i\in M}a_i x_i.$

We now use Lemma 2.5 in the usual way to select a witness $(\boldsymbol{a}_M, F_M, x^*_M, k_M)$ for each $M\in[\mathbb{N}]^\omega$ in a continuous way, where the set of all witnesses is given the discrete topology. We write $\boldsymbol{a}_M=(a_i^M)$ and $x_M=\sum_{i\in M}a_i^M x_i$.

We now carry out stabilizations. Fix $\epsilon>0$, and pass to an infnite subset N of $\mathbb{N}$ such that for all $P\in[N]^\omega$ we have

$$\sum_{i\in N\setminus P}\left|x^*_P(x_i)\right|\leq\epsilon, \tag{40}$$

and for all $L,M\in[N]^\omega$ we have $k_L=k_M$. The first property is achieved by Lemma 2.6, whereas the second uses infinite Ramsey theory. Observe that for all $L,M\in[N]^\omega$ we have

$$x^*_L(x_i)\geq\frac{1}{2}x^*_M(x_i) \qquad \text{whenever } i\in F_L\cap F_M. \tag{41}$$

We finally apply Theorem 2.2 with $n=1$ to find $L,M\in[N]^\omega$ such that

$$L\cap M=F_L\cap F_M=F_M. \tag{42}$$

We now estimate $x^*_L(x_M)$. On the one hand, using (42) followed by (41), (35), and (40), we have

$$\begin{aligned} x^*_L(x_M)=\sum_{i\in M}a_i^M x^*_L(x_i)=\sum_{i\in F_M}a_i^M x^*_L(x_i)&+\sum_{i\in M\setminus L}a_i^M x^*_L(x_i)\\ &\geq\frac{1}{2}\sum_{i\in F_M}a_i^M x^*_M(x_i)-4\epsilon\|x_M\|. \end{aligned}$$

On the other hand, property (39) applied to the witness of M gives

$$x^*_L(x_M)\leq\|x_M\|<\frac{2l}{A}\sum_{i\in F_M}a_i^M x^*_M(x_i)+\frac{\delta}{2A}.$$

The last two inequalities together with property (38) of the witness of M show that

$$\left(\frac{1}{2}-(1+4\epsilon)\frac{2l}{A}\right)\frac{\delta}{4l}\leq(1+4\epsilon)\frac{\delta}{2A}.$$

Since ϵ was arbitrary, it follows that $A \leq 8l$, as claimed. $\square$

Given a normalized, weakly null sequence (x_i) and $\delta \in (0,1]$ let $A((x_i),\delta)$ be the least real number A such that (x_i) is δ-convex-unconditional with constant A. Then define

$$A(\delta) = \sup_{(x_i)} \inf_{(y_i)\subset(x_i)} A((y_i),\delta),$$

where the supremum is taken over all normalized, weakly null sequences (x_i) and the infimum over all subsequences (y_i) of (x_i). Theorem 6.1 yields an upper bound of order $\log\big(1/\delta\big)$ on $A(\delta)$. We are now going to prove that the question whether $\sup_\delta A(\delta) < \infty$ is equivalent to Problem 1.2 using the function K' defined on page 26. As the proof shows the two problems are in some sense dual to each other.

Theorem 6.2. *For* $0 < \delta_1 < \delta \leq 1$ *we have*

(i) $A(\delta) \leq \frac{\delta}{\delta-\delta_1} K'(\delta_1)$, *and*
(ii) $K'(\delta) \leq A(\delta_1)$.

In particular $\sup_{\delta>0} A(\delta) = \sup_{\delta>0} K'(\delta)$.

Proof. We begin by proving (i). Fix $\epsilon \in (0,1]$. There is a normalized, weakly null sequence (x_i) such that $A((y_i),\delta) > A(\delta) - \epsilon$ for every subsequence (y_i) of (x_i). On the other hand, after passing to a subsequence if necessary, we may assume that $A((x_i),\delta) \leq A(\delta) + \epsilon$. Set $C = A(\delta)(\delta-\delta_1)/\delta - \epsilon(\delta+\delta_1)/\delta$. For each $M \in [\mathbb{N}]^\omega$ there is a triple $(\boldsymbol{a}, x^*, F)$, called a *witness for* M, where

$$\text{(43)} \quad \boldsymbol{a} = (a_i) \in c_{00}, \quad x^* \in B_{X^*}, \quad F \subset \{i \in M : |x^*(x_i)| \geq \delta_1\},$$

$$\text{(44)} \quad \sum_{i\in F} a_i x^*(x_i) > C\|x\|, \quad \text{where } x = \sum_{i\in M} a_i x_i.$$

Indeed, since $A((x_i)_{i\in M},\delta) > A(\delta) - \epsilon$, there exist $\boldsymbol{a} = (a_i) \in c_{00}$ and $E \in [M]^{<\omega}$ such that $\delta \sum_{i\in E} |a_i| \leq \big\| \sum_{i\in E} a_i x_i \big\|$, and $\big\| \sum_{i\in E} a_i x_i \big\| > (A(\delta) - \epsilon)\big\|x\big\|$, where $x = \sum_{i\in M} a_i x_i$. By homogeneity, we may assume that $\sum_{i\in E} |a_i| = 1$. Let $x^* \in B_{X^*}$ be a support functional for $\sum_{i\in E} a_i x_i$, and let $F = \{i \in E : |x^*(x_i)| \geq \delta_1\}$. An easy computation now shows that (44) holds.

We now use Lemma 2.5 in the usual way to obtain a continuous selection $M \mapsto (\boldsymbol{a}_M, x^*_M, F_M)$ of witnesses. We let $\boldsymbol{a}_M = (a_i^M)$ and $x_M = \sum_{i\in M} a_i^M x_i$.

Next we find infinite subsets $N_1 \supset N_2 \supset N_3$ of $\mathbb{N}$ as follows. First, there exists $N_1 \in [\mathbb{N}]^\omega$ such that $(x_i)_{i\in N_1}$ is a basic sequence with constant $1+\epsilon$. Then we apply Theorem 3.1 with $n=1$ to get $N_2 \in [N_1]^\omega$ such that $|a_i^M| \leq (1+\epsilon)\|x_M\|$ for all $M\in[N_2]^\omega$ and for all $i\in M$. Finally, by Lemma 2.6 there exists $N_3\in[N_2]^\omega$ such that $\sum_{i\in N_3\setminus M}|x^*_M(x_i)|<\epsilon$ for all $M\in[N_3]^\omega$.

After relabelling we may assume that $N_3=\mathbb{N}$. Let (e_i) be the unit vector basis of c_{00}, and for each $M\in[\mathbb{N}]^\omega$ set

$$t_M=\frac{1}{(1+\epsilon)^2\|x_M\|}\sum_{i\in M}a_i^M e_i,$$

which is an element of $[-1,1]^{\mathbb{N}}$ by the choice of N_2. We endow $[-1,1]^{\mathbb{N}}$ with the product topology and let S be the closure of $\{t_M : M\in[\mathbb{N}]^\omega\}\cup\{e_i : i\in\mathbb{N}\}$. Note that S is a compact metric space. For each $i\in\mathbb{N}$ let f_i be the i^{th} co-ordinate map. By the continuity of the choice of witnesses, S contains only sequences of finite support. Hence (f_i) is a normalized, weakly null sequence in $C(S)$. We claim that $K'((f_i)_{i\in M},\delta_1)\geq C/(1+\epsilon)^2$ for all $M\in[\mathbb{N}]^\omega$. Since ϵ was arbitrary, (i) follows from this claim.

Given $M\in[\mathbb{N}]^\omega$, set $n_M=\max F_M$, and let

$$f_M=\sum_{\substack{i\in M\\ i\leq n_M}}x^*_M(x_i)f_i.$$

For each $L\in[\mathbb{N}]^\omega$ we have

$$\begin{aligned}(1+\epsilon)^2\|x_L\||f_M(t_L)| &= \Big|\sum_{\substack{i\in L\cap M\\ i\leq n_M}}a_i^L x^*_M(x_i)\Big|\\ &\leq \Big|x^*_M\Big(\sum_{\substack{i\in L\\ i\leq n_M}}a_i^L x_i\Big)\Big|+\epsilon(1+\epsilon)\|x_L\|\leq(1+\epsilon)^2\|x_L\|\end{aligned}$$

by the choices of N_1,N_2 and N_3. It follows that $\|f_M\|\leq 1$. On the other hand, we have $F_M\subset\{i\in M : i\leq n_M,\ |x^*_M(x_i)|\geq\delta_1\}$ and

$$\begin{aligned}\Big\|\sum_{i\in F_M}x^*_M(x_i)f_i\Big\| &\geq \sum_{i\in F_M}x^*_M(x_i)f_i(t_M)\\ &= \frac{1}{(1+\epsilon)^2\|x_M\|}\sum_{i\in F_M}x^*_M(x_i)a_i^M>\frac{C}{(1+\epsilon)^2},\end{aligned}$$

as claimed.

To show (ii) fix $\epsilon\in(0,1]$ so that $\delta_1(1+3\epsilon)<\delta$, and let (x_i) be a normalized, weakly null sequence such that $K'((y_i),\delta)>K'(\delta)-\epsilon$ for every subsequence (y_i) of (x_i). So for each $M\in[\mathbb{N}]^\omega$ there is a triple $(\boldsymbol{a},E,x^*)$, called a *witness for M*, where

(45) $\boldsymbol{a}=(a_i)\in c_{00}, \quad E\subset\{i\in M: |a_i|\geq\delta\}, \quad x^*\in B_{X^*},$

(46) $\|x\|=1$, where $x=\sum_{i\in M} a_i x_i$,

(47) $\sum_{i\in E} a_i x^*(x_i) > K'(\delta)-\epsilon, \quad$ and $a_i x^*(x_i)>0$ for all $i\in E$.

As usual, we then have a continuous choice $M\mapsto(\boldsymbol{a}_M, E_M, x^*_M)$ of witnesses, and we let $\boldsymbol{a}_M=(a_i^M)$ and $x_M=\sum_{i\in M} a_i^M x_i$.

We now pass to infinite subsets $N_1\supset N_2\supset N_3$ of $\mathbb{N}$. First, we choose $N_1\in[\mathbb{N}]^\omega$ so that $(x_i)_{i\in N_1}$ is a basic sequence with constant $1+\epsilon$. Then we apply Theorem 3.1 with $n=1$ to find $N_2\in[N_1]^\omega$ such that we have $|a_i^M|\leq(1+\epsilon)\|x_M\|$ for all $M\in[N_2]^\omega$ and for all $i\in M$. Finally we use Lemma 2.6 in the usual way to obtain $N_3\in[N_2]^\omega$ so that $\sum_{i\in N_3\setminus M}|x^*_M(x_i)|<\epsilon$ for all $M\in[N_3]^\omega$.

We now relabel so that we can take $N_3=\mathbb{N}$. As before, we let (e_i) be the unit vector basis of c_{00}. We define S to be the closure in $[-1,1]^{\mathbb{N}}$ of the set

$$\Big\{\frac{1}{1+3\epsilon}\sum_{i\in M} a_i^M e_i : M\in[\mathbb{N}]^\omega\Big\}\cup\{e_i : i\in\mathbb{N}\}.$$

As before, it is easy to verify that S is a compact metric space containing only sequences of finite support, and that the sequence (f_i) of co-ordinate maps is a normalized, weakly null sequence in $C(S)$. We now show that $A((f_i)_{i\in M},\delta_1)\geq (K'(\delta)-\epsilon)(1+3\epsilon)^{-1}$ for all $M\in[\mathbb{N}]^\omega$.

Given $M\in[\mathbb{N}]^\omega$, set $n_M=\max E_M$, and let

$$f_M=\sum_{\substack{i\in M\\ i\leq n_M}} x^*_M(x_i) f_i.$$

For each $L\in[\mathbb{N}]^\omega$ we have

$$\Big|\sum_{\substack{i\in L\cap M\\ i\leq n_M}} a_i^L x^*_M(x_i)\Big|\leq\Big|x^*_M\Big(\sum_{\substack{i\in L\\ i\leq n_M}} a_i^L x_i\Big)\Big|+\epsilon(1+\epsilon)\leq 1+3\epsilon$$

by the choices of N_1, N_2 and N_3. It follows that $\|f_M\|\leq 1$. On the other hand, we have

$$\Big\|\sum_{i\in E_M} x^*_M(x_i) f_i\Big\|\geq\frac{1}{1+3\epsilon}\sum_{i\in E_M} x^*_M(x_i)a_i^M\geq\delta_1\sum_{i\in E_M}|x^*_M(x_i)|,$$

as well as

$$\Big\|\sum_{i\in E_M} x^*_M(x_i) f_i\Big\|\geq\frac{1}{1+3\epsilon}\sum_{i\in E_M} x^*_M(x_i)a_i^M>\frac{K'(\delta)-\epsilon}{1+3\epsilon}\,\|f_M\|,$$

which proves the claim. $\square$

7. Unconditionality in $C(S)$ spaces and duality

We now turn to questions on finding unconditional basic sequences in spaces of continuous functions on a compact, Hausdorff space. We will then relate these to problems considered so far. We start by stating a result of Rosenthal.

Theorem 7.1. *For any compact, Hausdorff space S, every weakly null sequence of (non-zero) indicator functions in $C(S)$ has an unconditional subsequence with constant* 1*.*

In [4] this is presented as a consequence of a combinatorial lemma. Here we prove a more general version of that, and obtain a more general version of Theorem 7.1. Before stating it we need some notation. Let $k\in\mathbb{N}$ and $M\in[\mathbb{N}]^\omega$. Given $\boldsymbol{a}=(a_i)_{i\in M}$ and $\boldsymbol{b}=(b_i)_{i\in M}$ in $\{0,1,\dots,k\}^M$, we write $\boldsymbol{a}\subset\boldsymbol{b}$ if for all $i\in M$ either $a_i=0$ or $a_i=b_i$. Given $j\in\{1,\dots,k\}$, we write $\boldsymbol{a}\subset_j\boldsymbol{b}$ if for all $i\in M$ either $a_i=0$ or $a_i=b_i=j$. A family $\mathcal{F}\subset\{0,1,\dots,k\}^M$ is *hereditary* if $\boldsymbol{a}\in\mathcal{F}$ whenever $\boldsymbol{b}\in\mathcal{F}$ and $\boldsymbol{a}\subset\boldsymbol{b}$, and is *weakly hereditary* if $\boldsymbol{a}\in\mathcal{F}$ whenever $\boldsymbol{b}\in\mathcal{F}$ and there exists $j\in\{1,\dots,k\}$ such that $\boldsymbol{a}\subset_j\boldsymbol{b}$. Given $L\in[M]^\omega$, we denote by $\mathcal{F}_L$ the set of restrictions to L of elements of $\mathcal{F}$. Note that $\mathcal{F}_L\subset\{0,1,\dots,k\}^L$.

Lemma 7.2. *Let $k\in\mathbb{N}$ and $\mathcal{F}\subset\{0,1,\dots,k\}^{\mathbb{N}}$ be a compact family of sequences of finite support. Then there exists $M\in[\mathbb{N}]^\omega$ such that $\mathcal{F}_M$ is weakly hereditary.*

Proof. We argue by contradiction. Assuming that the statement is false, for each $M\in[\mathbb{N}]^\omega$ we can find a quadruple $(\boldsymbol{a},\boldsymbol{b},j,K)$ that we shall call a *witness for* M, where

(48) $\boldsymbol{a}\in\{0,1,\dots,k\}^{\mathbb{N}},\quad \boldsymbol{b}\in\mathcal{F},\quad j\in\{1,\dots,k\},\quad K\in\mathbb{N};$

(49) $\operatorname{supp}(\boldsymbol{a})\subset M,\quad \boldsymbol{a}\subset_j\boldsymbol{b},\quad K>\max\operatorname{supp}(\boldsymbol{b});$

(50) if $a_i=c_i$ for all $i\in M$ with $i\le K$, then $\boldsymbol{c}\notin\mathcal{F}$.

Indeed, the assumption that $\mathcal{F}_M$ is not weakly hereditary implies the existence of $\boldsymbol{a},\boldsymbol{b},j$ as in (48) such that $\operatorname{supp}(\boldsymbol{a})\subset M$, $\boldsymbol{a}\subset_j\boldsymbol{b}$ and there is no $\boldsymbol{c}\in\mathcal{F}$ such that the restrictions to M of $\boldsymbol{a}$ and $\boldsymbol{c}$ are identical. The existence of a suitable K now follows easily from the compactness of $\mathcal{F}$.

Let Ω denote the set of all witnesses of all infinite subsets of $\mathbb{N}$. For $r\in\mathbb{N}$ let Ω_r be the set of elements $(\boldsymbol{a},\boldsymbol{b},j,K)\in\Omega$ for which $K\le r$. The conditions of Lemma 2.5 are now easily verified (which is why we needed to introduce the parameter K). So there is a continuous selection $\phi\colon[\mathbb{N}]^\omega\to\Omega$ of witnesses. Let $\phi(M)=(\boldsymbol{a}_M,\boldsymbol{b}_M,j_M,K_M)$, where $\boldsymbol{a}_M=(a_i^M)$ and $\boldsymbol{b}_M=(b_i^M)$ for each $M\in[\mathbb{N}]^\omega$.

The continuity of ϕ and the compactness of $\mathcal{F}$ imply that the function $M\mapsto\boldsymbol{b}_M\colon[\mathbb{N}]^\omega\to c_0$ is continuous and its image has compact closure (in the topology of pointwise convergence). So applying Lemma 2.6 with $\epsilon = 1/2$, say, we find $N_1 \in [\mathbb{N}]^\omega$ such that

$$N_1 \cap \operatorname{supp}(\boldsymbol{b}_L) \subset L \qquad \text{for all } L \in [N_1]^\omega. \tag{51}$$

An easy application of infinite Ramsey theory then gives $j \in \{1, \dots, k\}$ and $N_2 \in [N_1]^\omega$ such that $j_M = j$ for all $M \in [N_2]^\omega$.

To conclude the proof we apply the Matching Lemma with $n=1$ to the function $M \mapsto \operatorname{supp}(\boldsymbol{a}_M)$ to find $L, M \in [N_2]^\omega$ such that

$$L \cap M = \operatorname{supp}(\boldsymbol{a}_L) \cap \operatorname{supp}(\boldsymbol{a}_M) = \operatorname{supp}(\boldsymbol{a}_M).$$

Now if $i \in \operatorname{supp}(\boldsymbol{a}_M)$, then $a_i^M = a_i^L = b_i^L = j$ by property (49) of a witness and by the choice of N_2. On the other hand, if $i \in M \setminus \operatorname{supp}(\boldsymbol{a}_M)$, then $i \notin L$, and hence by (51) we have $i \notin \operatorname{supp}(\boldsymbol{b}_L)$, so $a_i^M = b_i^L = 0$. We have shown that the restrictions to M of $\boldsymbol{a}_M$ and the element $\boldsymbol{b}_L$ of $\mathcal{F}$ are identical which gives the required contradiction. $\square$

Theorem 7.3. *For all $\delta \in (0,1]$ there is a constant L^* such that for any compact, Hausdorff space S, if (f_i) is a normalized, weakly null sequence in $C(S)$ with $|f_i(t)| \in \{0\} \cup [\delta, 1]$ for all $t \in S$ and $i \in \mathbb{N}$, then (f_i) has an unconditional subsequence with constant L^*. Moreover, $L^* \le 6 \log_2 \big(1/\delta\big)$ for $\delta < 1/4$.*

Proof. For $\delta \in (0,1]$ let $k = \lfloor \log_2(1/\delta) \rfloor + 1$. Let $I_0 = \{0\}$ and let $I_1, \dots, I_k$ be closed intervals covering $[\delta, 1]$ such that $\max I_j \le 2 \min I_j$ for each $j = 1, \dots, k$. Furthermore, let $I_{j+k} = -I_j$ for $j = 1, \dots, k$. Let S be a compact, Hausdorff space and (f_i) be a normalized, weakly null sequence in $C(S)$ with $|f_i(t)| \in \{0\} \cup [\delta, 1]$ for all $t \in S$ and $i \in \mathbb{N}$. Let $\mathcal{F}$ be the collection of all $\boldsymbol{c} \in \{0, 1, \dots, 2k\}^{\mathbb{N}}$ for which there exists $t \in S$ with $f_i(t) \in I_{c_i}$ for all $i \in \mathbb{N}$. Note that $\mathcal{F}$ is a compact subset of $\{0, 1, \dots, 2k\}^{\mathbb{N}}$ consisting of sequences of finite support. By Lemma 7.2 there exists $M \in [\mathbb{N}]^\omega$ such that $\mathcal{F}_M$ is weakly hereditary. We show that the sequence $(f_i)_{i \in M}$ is unconditional with constant $L^* = 4k$.

Fix $\boldsymbol{a} = (a_i) \in c_{00}$ and $E \in [M]^{<\omega}$. Choose $t \in S$ such that

$$\Big\| \sum_{i \in E} a_i f_i \Big\| = \Big| \sum_{i \in E} a_i f_i(t) \Big|.$$

Replacing $\boldsymbol{a}$ by $-\boldsymbol{a}$ if necessary, we may assume that

$$\Big| \sum_{i \in E} a_i f_i(t) \Big| \le \sum_{i \in F} a_i f_i(t),$$

where $F=\{i\in E\,:\,a_if_i(t)>0\}$. Now choose $\boldsymbol{c}\in\mathcal{F}$ such that $f_i(t)\in I_{c_i}$ for all $i\in\mathbb{N}$. Note that $c_i\neq 0$ for any $i\in F$, and so

$$\sum_{i\in F}a_if_i(t)\leq 2k\sum_{i\in F_j}a_if_i(t)$$

for some $j\in\{1,\dots,2k\}$, where $F_j=\{i\in F\,:\,c_i=j\}$. Finally, since $\mathcal{F}_M$ is weakly hereditary, there exists $\boldsymbol{c}'\in\mathcal{F}$ such that $c'_i=c_i=j$ for all $i\in F_j$, and $c'_i=0$ for all $i\in M\setminus F_j$. Let $t'\in S$ satisfy $f_i(t')\in I_{c'_i}$ for all $i\in\mathbb{N}$. We then have

$$\sum_{i\in F_j}a_if_i(t)\leq 2\sum_{i\in M}a_if_i(t')\leq 2\Big\|\sum_{i\in M}a_if_i\Big\|.$$

This completes the proof of our claim. □

Remarks. 1. If (f_i) is a weakly null sequence of (non-zero) indicator functions, then in the proof above we need only to work with two intervals $I_0=\{0\}$ and $I_1=\{1\}$. This way we do not get the factor of 2 at either of the two places where it occurs above, and so we obtain a proof of Theorem 7.1. We also mention here a quantitative version of Rosenthal's result due to Gasparis, Odell and Wahl [12]: if (f_i) is a weakly null sequence of (non-zero) indicator functions, then there exists a countable ordinal α and a subsequence (g_i) of (f_i) which is equivalent to a subsequence of the unit vector basis of the generalized Schreier space X^α.
2. Lemma 7.2 and Theorem 7.3 were also proved by Arvanitakis (he uses slightly different language and method). In [6, Remark 2.1] he effectively asks if weakly hereditary can be replaced by hereditary in Lemma 7.2. It is not hard to see that if that was possible, then the proof of Theorem 7.3 would give a constant L^* independent of δ. In turn, by Theorem 7.5 below, this would yield a positive solution to the c_0-problem. The following simple example shows that Lemma 7.2 cannot be strengthened in this way even for $k=2$. For each $M=\{m_1<m_2<\dots\}\in[\mathbb{N}]^\omega$ define $c_M\in\{0,1,2\}^{\mathbb{N}}$ by letting

$$c_M(m_i)=\begin{cases}2 & \text{if } i=1\\ 1 & \text{if } 2\leq i\leq m_1+1\\ 2 & \text{if } m_1+1<i\leq m_2+1\end{cases}$$

and c_M is zero elsewhere. Now let $\mathcal{F}$ be the set of all $c\in\{0,1,2\}^{\mathbb{N}}$ such that there exist $M\in[\mathbb{N}]^\omega$ and $n\in\mathbb{N}$ such that $c(i)=c_M(i)$ for $i=1,\dots,n$ and $c(i)=0$ for all $i>n$ — we denote this c by $c_{M,n}$. Then $\mathcal{F}$ is a compact family of sequences of finite support. To see that $\mathcal{F}_L$ is not hereditary for any $L=\{l_1<l_2<\dots\}\in[\mathbb{N}]^\omega$ consider $c,c'\in\{0,1,2\}^L$ defined as follows: $c'(l_1)=0$, $c(l_1)=c_L(l_1)$ and $c'(l_i)=c(l_i)=c_L(l_i)$

for all $i \geq 2$. Then $c \in \mathcal{F}_L$, $c' \subset c$, but given $M = \{m_1 < m_2 < \dots\} \in [\mathbb{N}]^\omega$, there is no $n \in \mathbb{N}$ such that c' is the restriction to L of $c_{M,n}$ (consider the cases $m_2 < l_2$, $m_2 = l_2$ and $m_2 > l_2$).

We now prove a more general result of which Theorem 7.3 is an immediate consequence.

Theorem 7.4. *For all $\delta \in (0,1]$ there is a constant K^* such that for any compact, Hausdorff space S, every normalized, weakly null sequence (f_i) in $C(S)$ has a subsequence (g_i) such that for all $t \in S$ and $E \subset \{i \in \mathbb{N} : |g_i(t)| \geq \delta\}$ we have*

$$\Big|\sum_{i \in E} a_i g_i(t)\Big| \leq K^* \Big\|\sum_{i=1}^{\infty} a_i g_i\Big\| \qquad \text{for all } (a_i) \in \mathrm{c}_{00}.$$

Moreover, $K^* \leq 6 \log_2 \big(1/\delta\big)$ for $\delta < 1/4$.

Proof. Fix $\delta \in (0,1]$ and $K^* \in [1,\infty)$. Assume that S is a compact, Hausdorff space, and (f_i) is a normalized, weakly null sequence in $C(S)$ that has no subsequence satisfying the statement of the theorem. We will show that $K^* \leq 4k$, where $k = \lfloor \log_2(1/\delta) \rfloor + 1$.

Let $I_1, \dots, I_k$ be closed intervals covering $[\delta, 1]$ such that $\max I_j \leq 2 \min I_j$ for each $j = 1, \dots, k$. Furthermore, let $I_{j+k} = -I_j$ for $j = 1, \dots, k$. For every $M \in [\mathbb{N}]^\omega$ there is a *witness* $(t, \boldsymbol{a}, j, F)$ to the failure of the subsequence $(f_i)_{i \in M}$, where

(52) $t \in S, \quad \boldsymbol{a} = (a_i) \in \mathrm{c}_{00}, \quad j \in \{1, \dots, 2k\},$

$F \subset \{i \in M : f_i(t) \in I_j,\ a_i f_i(t) > 0\};$

(53) $\|f\| = 1$, where $f = \sum_{i \in M} a_i f_i;$

(54) $2k \sum_{i \in F} a_i f_i(t) > K^*.$

We now use Lemma 2.5 to get a continuous selection $M \mapsto (t_M, \boldsymbol{a}_M, j_M, F_M)$ of witnesses. Let $\boldsymbol{a}_M = (a_i^M)$ and $f_M = \sum_{i \in M} a_i^M f_i$ for each $M \in [\mathbb{N}]^\omega$.

As usual, the next phase of the proof is stabilization. Find $N_1 \in [\mathbb{N}]^\omega$ such that $(f_i)_{i \in N_1}$ is a basic sequence with constant 2, and so $|a_i^M| \leq 4$ for all $i \in M$ and for all $M \in [N_1]^\omega$. Then pass to $N_2 \in [N_1]^\omega$ such that $j_L = j_M$ for all $L, M \in [N_2]^\omega$, which in particular implies that $f_i(t_M)$ and $f_i(t_L)$ have the same sign, and differ by a factor of at most 2 for all $i \in F_L \cap F_M$. Finally, we fix $\epsilon > 0$ and use Lemma 2.6 to obtain $N_3 \in [N_2]^\omega$ such that $\sum_{i \in N_3 \setminus P} |f_i(t_P)| < \epsilon$ for all $P \in [N_3]^\omega$.

The Matching Lemma applied with $n=1$ now yields $L,M\in[N_3]^\omega$ such that $L\cap M=F_L\cap F_M=F_M$. Then

$$\begin{aligned}|f_M(t_L)| &= \Big|\sum_{i\in M} a_i^M f_i(t_L)\Big|\\ &\geq \sum_{i\in F_M} a_i^M f_i(t_L)-4\epsilon\\ &\geq \frac{1}{2}\sum_{i\in F_M} a_i^M f_i(t_M)-4\epsilon\\ &\geq \frac{K}{4k}-4\epsilon.\end{aligned}$$

On the other hand, $|f_M(t_L)|\leq\|f_M\|=1$, and hence $K\leq 4k(1+4\epsilon)$. □

We will now establish a relationship between Theorem 7.3, which is a result about finding unconditional subsequences, and the constant L' (defined on page 26), which comes from a certain form of partial unconditionality. We will also show the close connection between Theorem 7.4 and Problem 1.2. First we need to introduce some appropriate constants, and then we will express these relationships in Theorem 7.5 below.

For a basic sequence (x_i) in a Banach space let $C(x_i)$ be the least real number C such that (x_i) is unconditional with constant C. Then for each $\delta\in(0,1]$ we define

$$L^*(\delta)=\sup_{S,(f_i)}\ \inf_{(g_i)\subset(f_i)} C(g_i),$$

where the supremum is taken over all compact, Hausdorff spaces S and over all normalized, weakly null sequences (f_i) in $C(S)$ with $|f_i(t)|\in\{0\}\cup[\delta,1]$ for all $t\in S$ and $i\in\mathbb{N}$, and the infimum is taken over subsequences (g_i) of (f_i). Theorem 7.3 above claims that $L^*(\delta)$ is finite and of order $\log\big(1/\delta\big)$.

Given $\delta\in(0,1]$, and a normalized, weakly null sequence (f_i) in $C(S)$ with S a compact, Hausdorff space, we define $K^*((f_i),\delta)$ to be the least real number K^* such that whenever $t\in S$ and $E\subset\{i\in\mathbb{N}:\,|f_i(t)|\geq\delta\}$, we have

$$\Big|\sum_{i\in E} a_i f_i(t)\Big|\leq K^*\Big\|\sum_{i=1}^{\infty} a_i f_i\Big\| \qquad \text{for all } (a_i)\in c_{00}.$$

We then set

$$K^*(\delta)=\sup_{S,(f_i)}\ \inf_{(g_i)\subset(f_i)} K^*((g_i),\delta),$$

where the supremum is over all compact, Hausdorff spaces S and all normalized, weakly null sequences (f_i) in $C(S)$, and the infimum is over all subsequences (g_i) of (f_i). Note that by Theorem 7.4 above $K^*(\delta)$ is finite and of order $\log\big(1/\delta\big)$.

Theorem 7.5. *For all* $0<\delta'<\delta\leq 1$ *we have* $K^*(\delta)\leq K'(\delta)\leq K^*(\delta')$ *and* $L^*(\delta)\leq L'(\delta)\leq L^*(\delta')$.

Proof. We first show that $K^*(\delta)\leq K'(\delta)$. Fix $\epsilon\in(0,1]$. There is a compact Hausdorff space S and a normalized, weakly null sequence (f_i) in $C(S)$ such that $K^*((f_i)_{i\in M},\delta)>K^*(\delta)-\epsilon$ for all $M\in[\mathbb{N}]^\omega$. So for each $M\in[\mathbb{N}]^\omega$ there is a *witness* $(t,E,\boldsymbol{a})$ for M, where

(55) $t\in S,\quad E\subset\{i\in M:\,|f_i(t)|\geq\delta\},\quad \boldsymbol{a}=(a_i)\in c_{00};$

(56) $\|f\|=1,\quad \text{where } f=\displaystyle\sum_{i\in M}a_if_i;$

(57) $\Big|\displaystyle\sum_{i\in E}a_if_i(t)\Big|>K^*(\delta)-\epsilon.$

We now proceed as usual. We make a continuous choice $M\mapsto(t_M,E_M,\boldsymbol{a}_M)$ of witnesses, and let $\boldsymbol{a}_M=(a_i^M)$ and $f_M=\sum_{i\in M}a_i^Mf_i$. We then find $N_1\in[\mathbb{N}]^\omega$ such that $(f_i)_{i\in N_1}$ is a basic sequence with constant $1+\epsilon$. By Theorem 3.1 there exists $N_2\in[N_1]^\omega$ such that $|a_i^M|\leq 1+\epsilon$ for all $i\in M$ and for all $M\in[N_2]^\omega$. Finally, we pass to a further infinite subset N_3 of N_2 such that $\sum_{i\in N_3\setminus P}|f_i(t_P)|<\epsilon$ for all $P\in[N_3]^\omega$.

After relabelling, if necessary, we may assume that $N_3=\mathbb{N}$. We define a norm on c_{00} by letting

$$\|(b_i)\|=\sup_i|b_i|\vee\frac{1}{(1+\epsilon)^2}\sup\Big\{\Big|\sum_{i\in M}b_ia_i^M\Big|:\,M\in[\mathbb{N}]^\omega\Big\}$$

for each $(b_i)\in c_{00}$. Let X be the completion of the resulting normed space. It is easy to check that the unit vector basis (e_i) of c_{00} is a normalized, weakly null sequence in X. Indeed, the continuity of the selection of witnesses implies that the closure of $\big\{\sum_{i\in M}a_i^Me_i:\,M\in[\mathbb{N}]^\omega\big\}\cup\{e_i:\,i\in\mathbb{N}\}$ in the topology of pointwise convergence contains only finitely supported sequences. We will now show that $K'((y_i),\delta)>K^*(\delta)-\epsilon$ for any subsequence (y_i) of (e_i). This then proves the inequality $K^*(\delta)\leq K'(\delta)$.

Fix $M\in[\mathbb{N}]^\omega$, and set $n_M=\max E_M$ and

$$x_M=\sum_{\substack{i\in M\\ i\leq n_M}}f_i(t_M)e_i.$$

For each $L\in[\mathbb{N}]^\omega$ we have

$$\begin{aligned}\Big|\sum_{\substack{i\in L\cap M\\ i\le n_M}} f_i(t_M)a_i^L\Big| &\le \Big|\sum_{\substack{i\in L\\ i\le n_M}} f_i(t_M)a_i^L\Big| + (1+\epsilon)\sum_{i\in L\setminus M}|f_i(t_M)|\\ &\le (1+\epsilon)\|f_L\| + (1+\epsilon)\epsilon = (1+\epsilon)^2.\end{aligned}$$

It follows that $\|x_M\|\le 1$. On the other hand, on E_M the coefficients of x_M are at least δ and

$$\Big\|\sum_{i\in E_M} f_i(t_M)e_i\Big\| \ge \Big|\sum_{i\in E_M} f_i(t_M)a_i^M\Big| > K^*(\delta)-\epsilon.$$

We now show that $K'(\delta)\le K^*(\delta')$ whenever $0<\delta'<\delta\le 1$. Fix $\epsilon\in(0,1]$ such that $(1+\epsilon)\delta'<\delta$. Let (x_i) be a normalized, weakly null sequence with $K'((y_i),\delta)>K'(\delta)-\epsilon$ for every subsequence (y_i) of (x_i). So for each $M\in[\mathbb{N}]^\omega$ there is a *witness* $(\boldsymbol{a},E,x^*)$ *of* M, where

(58) $\boldsymbol{a}=(a_i)\in c_{00},\quad E\subset\{i\in M:\,|a_i|\ge\delta\},\quad x^*\in B_{X^*}$;

(59) $\|x\|=1,\quad$ where $x=\sum_{i\in M} a_ix_i$;

(60) $\sum_{i\in E} a_ix^*(x_i) > K'(\delta)-\epsilon.$

Let $M\mapsto(\boldsymbol{a}_M,E_M,x^*_M)$ be a continuous selection of witnesses, and let $\boldsymbol{a}_M=(a_i^M)$ and $x_M=\sum_{i\in M}a_i^Mx_i$. Choose $N_1\in[\mathbb{N}]^\omega$ such that $(x_i)_{i\in N_1}$ is a basic sequence with constant $1+\epsilon$. Use Theorem 3.1 to find $N_2\in[N_1]^\omega$ so that $|a_i^M|\le 1+\epsilon$ for all $i\in M$ and $M\in[N_2]^\omega$. Finally, by Lemma 2.6 there exists $N_3\in[N_2]^\omega$ such that $\sum_{i\in N_3\setminus P}|x^*_P(x_i)|<\epsilon$ for all $P\in[N_3]^\omega$.

Relabel so that we can take $N_3=\mathbb{N}$, and set $t_M=\frac{1}{1+\epsilon}\sum_{i\in M}a_i^Me_i$ for each $M\in[\mathbb{N}]^\omega$, where (e_i) is the unit vector basis of c_{00}. Let S be the closure of the set $\{t_M:\,M\in[\mathbb{N}]^\omega\}\cup\{e_i:\,i\in\mathbb{N}\}$ in the product space $[-1,1]^{\mathbb{N}}$. As before, it is easy to verify that S consists only of finitely supported sequences, and hence the sequence (f_i) of co-ordinate maps is a normalized, weakly null sequence in $C(S)$. We will show that

$$K^*((g_i),\delta') > \frac{K'(\delta)-\epsilon}{(1+\epsilon)^2}$$

for every subsequence (g_i) of (f_i), which then implies that $K^*(\delta')\ge K'(\delta)$.

Fix $M\in[\mathbb{N}]^\omega$ and let $n_M=\max\operatorname{supp}(\boldsymbol{a}_M)$ and

$$f_M=\sum_{\substack{i\in M\\ i\le n_M}} x^*_M(x_i)f_i.$$

For each $L\in[\mathbb{N}]^\omega$ we have

$$\begin{aligned}(1+\epsilon)|f_M(t_L)| &= \Big|\sum_{\substack{i\in L\cap M\\ i\leq n_M}} x_M^*(x_i)a_i^L\Big| \\ &\leq \Big|\sum_{\substack{i\in L\\ i\leq n_M}} x_M^*(x_i)a_i^L\Big| + (1+\epsilon)\sum_{i\in L\setminus M}|x_M^*(x_i)| \\ &\leq (1+\epsilon)\|x_L\| + (1+\epsilon)\epsilon \leq (1+\epsilon)^2.\end{aligned}$$

It follows that $\|f_M\|\leq 1+\epsilon$. On the other hand, we have

$$|f_i(t_M)| = |a_i^M|/(1+\epsilon) > \delta' \qquad \text{for all } i\in E_M,$$

and moreover

$$\sum_{\substack{i\in E_M\\ i\leq n_M}} x_M^*(x_i)f_i(t_M) = \sum_{i\in E_M} x_M^*(x_i)a_i^M/(1+\epsilon) > \frac{K'(\delta)-\epsilon}{(1+\epsilon)^2}\|f_M\|.$$

This completes the proof of the inequalities involving K' and K^*. The argument for the functions L' and L^* is similar and is omitted. □

Recall that if (x_i) is a normalized, weakly null sequence with spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon>0$ and for any $\delta\in(0,1]$ there is a δ-near-unconditional subsequence of (x_i) with constant $1+\epsilon$. There are dual versions of this corresponding to Theorems 7.3 and 7.4 above. For example, for any compact, Hausdorff space S and for any $\delta\in(0,1]$, if (f_i) is a normalized, weakly null sequence in $C(S)$ with $|f_i(t)|\in\{0\}\cup[\delta,1]$ for all $t\in S$ and $i\in\mathbb{N}$, and (f_i) has spreading model not equivalent to the unit vector basis of ℓ_1, then for any $\epsilon>0$ there is a subsequence of (f_i) that is unconditional with constant $1+\epsilon$. The proof (which we omit here) uses a similar argument to that of [10, Theorem 5.4].

8. The combinatorics of patterns and resolutions

In this section we consider combinatorial structures that arise in our approach to Problem 1.2. We begin by setting up witnesses for the constant $K'(\delta)$ (c.f. Definition 4.1). The notation will be used throughout this section. We fix $\delta\in(0,1]$, set $k=\lfloor\log_2(1/\delta)\rfloor+1$, and choose $\epsilon\in(0,1)$ so that $2^k\delta>1+\epsilon$. We then select closed intervals $I_1,\dots,I_k$ covering $[\delta,1+\epsilon]$ so that $\max I_j\leq 2\min I_j$ for each $j=1,\dots,k$. By the definition of $K'(\delta)$ there is a normalized, weakly null sequence (x_i) in some Banach space X such that $K'((y_i),\delta)>\frac{1}{2}K'(\delta)$ for every subsequence (y_i) of (x_i). After passing to a subsequence if necessary we can assume, as usual, that

(61) (x_i) is a basic sequence with constant $1+\epsilon$,

(62) $\sup_i |a_i| \le (1+\epsilon)\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\|$ for all $(a_i)\in c_{00}$.

Recall that the latter property is achieved using Theorem 3.1. We now make a continuous selection $M \mapsto (\boldsymbol{a}_M, x_M^*, F_M)$ of witnesses in the usual manner using Lemma 2.5, where

(63) $\boldsymbol{a}_M = (a_i^M)\in c_{00}, \quad x_M^* \in B_{X^*}, \quad F_M \subset \{i\in M : a_i^M \ge \delta,\ x_M^*(x_i)>0\}$,

(64) $\|x_M\| = 1$, where $x_M = \sum_{i\in M} a_i^M x_i$,

(65) $\sum_{i\in F_M} a_i^M x_M^*(x_i) > K'(\delta)/4$.

Note that $|a_i^M|\in[\delta, 1+\epsilon]$ for all $M\in[\mathbb{N}]^\omega$ and for all $i\in F_M$. For each $M\in[\mathbb{N}]^\omega$ let us define $\boldsymbol{c}_M = (c_i^M)\in\{0,1,\dots,k\}^{\mathbb{N}}$ by letting c_i^M be the least $j\in\{1,\dots,k\}$ such that $a_i^M\in I_j$ if $i\in F_M$, and letting $c_i^M = 0$ otherwise. Set $F_j^M = \{i\in\mathbb{N} : c_i^M = j\}$ for each $j=1,\dots,k$. Note that

(66) $F_1^M,\dots,F_k^M$ are pairwise disjoint, finite subsets of M with $F_M = \bigcup_{j=1}^k F_j^M$,

(67) for each $j=1,\dots,k$ the function $M\mapsto F_j^M : [\mathbb{N}]^\omega \to [\mathbb{N}]^{<\omega}$ is continuous.

Note that we have $\operatorname{osc}(\boldsymbol{a}_M, F_j^M)\le 2$ for all $M\in[\mathbb{N}]^\omega$ and for each $j=1,\dots,k$. Moreover, for any two infinite subsets L, M of $\mathbb{N}$ we have

$$\frac{a_i^M}{a_i^L} \ge \frac{1}{2} \qquad \text{for all } i\in F_j^L\cap F_j^M,\ j=1,\dots,k. \tag{68}$$

Using the usual Ramsey type arguments (Lemma 2.6 and the infinite Ramsey theorem) and relabeling, if necessary, we may assume the following stabilizations.

(69) $\sum_{i\notin M} |x_M^*(x_i)| < \epsilon$ for all $M\in[\mathbb{N}]^\omega$;

(70) for each $j=1,\dots,k$ there exists w_j such that for all $M\in[\mathbb{N}]^\omega$ we have $w_j \le \sum_{i\in F_j^M} a_i^M x_M^*(x_i) \le w_j + \epsilon/k$.

Observe that (65) and (70) give

(71) $\sum_{j=1}^k w_j > \dfrac{K'(\delta)}{4} - \epsilon$.

We now give a simple necessary and sufficient condition for a positive answer to Problem 1.2.

Proposition 8.1. *We have* $\sup_{\delta>0} K'(\delta)<\infty$ *if and only if there is a constant* c *such that for all* $\delta\in(0,1]$ *whenever* (x_i) *is a normalized, weakly null sequence in a Banach space and* $M\mapsto(\boldsymbol{a}_M,x^*_M,F_M)$ *is a continuous selection of witnesses so that* (61)–(65) *and* (69) *hold, then there exist infinite subsets* L,M *of* $\mathbb{N}$ *such that* $|x^*_L(x_M)|\geq cK'(\delta)$.

Proof. Sufficiency is clear: for any $\delta\in(0,1]$ there is a normalized, weakly null sequence (x_i) and a continuous selection $M\mapsto(\boldsymbol{a}_M,x^*_M,F_M)$ of witnesses so that (61)–(65) and (69) hold. The assumption then gives $K'(\delta)\leq 1/c$.

Now assume that $K'=\sup_{\delta>0}K'(\delta)$ is finite. We show that the condition is necessary with $c=\frac{1}{16K'}$. Let $\delta\in(0,1]$ and assume that we are given a normalized, weakly null sequence (x_i) and a continuous selection $M\mapsto(\boldsymbol{a}_M,x^*_M,F_M)$ of witnesses so that (61)–(65) and (69) hold. Set $t=\frac{K'(\delta)}{16K'}$. For $\boldsymbol{b}=(b_i)\in c_{00}$ define

$$|||\boldsymbol{b}|||=t\|\boldsymbol{b}\|_{\ell_\infty}\vee\sup\Big\{\Big|\sum_{i=1}^\infty b_i x^*_L(x_i)\Big|:\,L\in[\mathbb{N}]^\omega\Big\}.$$

Let Z be the completion of c_{00} in the norm $|||\cdot|||$. The unit vector basis (e_i) of c_{00} is a semi-normalized, weakly null sequence of Z. So by the definition of $K'(t\delta)$ there is an infinite subset M of $\mathbb{N}$ such that $K'((\hat{e}_i)_{i\in M},t\delta)<2K'(t\delta)$, say, where $\hat{e}_i=\frac{e_i}{|||e_i|||}$ for all $i\in\mathbb{N}$. From (65) we get

$$\Big|\Big|\Big|\sum_{i\in F_M}a_i^M e_i\Big|\Big|\Big|\geq\Big|\sum_{i\in F_M}a_i^M x^*_M(x_i)\Big|>K'(\delta)/4.$$

Now let $\boldsymbol{b}_M=\sum_{i\in M}a_i^M e_i$. By (62) we have $\|\boldsymbol{b}_M\|_{\ell_\infty}\leq 1+\epsilon$, which in turn gives $|||\boldsymbol{b}_M|||\leq 1$ since x^*_L has norm at most one for all $L\in[\mathbb{N}]^\omega$. Now since $\big|a_i^M|||e_i|||\big|\geq t\delta$ for each $i\in F_M$, we have

$$\Big|\Big|\Big|\sum_{i\in F_M}a_i^M e_i\Big|\Big|\Big|=\Big|\Big|\Big|\sum_{i\in F_M}a_i^M|||e_i|||\hat{e}_i\Big|\Big|\Big|\leq 2K'(t\delta)|||\boldsymbol{b}_M|||.$$

We can now conclude that

$$|||\boldsymbol{b}_M|||>\frac{K'(\delta)}{8K'(t\delta)}>t\|\boldsymbol{b}_M\|_{\ell_\infty}.$$

Hence there exists $L\in[\mathbb{N}]^\omega$ such that

$$|x^*_L(x_M)|=\Big|\sum_{i\in M}a_i^M x^*_L(x_i)\Big|>\frac{1}{2}|||\boldsymbol{b}_M|||\geq cK'(\delta).$$

□

A selection of witnesses as defined in (63)–(65) associates to each $M\in[\mathbb{N}]^\omega$ a certain combinatorial data that is made up of two parts. One part is the sequence $(c_i^M)_{i\in F_M}$ in the set $\{1,\dots,k\}$, which is a discretized version of the coefficients $(a_i^M)_{i\in F_M}$ of the vector x_M. Equivalently, this part can also be viewed as the partition $(F_j^M)_{j=1}^k$ of F_M. The other part is the sequence $\big(x_M^*(x_i)\big)_{i\in F_M}$ of dual coefficients. To solve Problem 1.2 in the affirmative we would like to show the existence of $L,M\in[\mathbb{N}]^\omega$ whose combinatorial data "match" in a suitable way to give the necessary and sufficient condition of Proposition 8.1. For example, if we could assume that the sets $F_1^M,\dots,F_k^M$ are successive for all $M\in[\mathbb{N}]^\omega$, then the Matching Lemma would provide suitable sets L and M. Indeed, we can generalize this as follows.

Proposition 8.2. *The following is a sufficient condition for* $\sup_{\delta>0}K'(\delta)<\infty$. *There exists a constant c such that for all $k\in\mathbb{N}$ and for all positive real numbers $p_1,\dots,p_k$ with $\sum_{j=1}^k p_j=1$ if for all $M\in[\mathbb{N}]^\omega$ we are given finite subsets $F_1^M,\dots,F_k^M$ of M such that* (66) *and* (67) *hold, then there exist $L,M\in[\mathbb{N}]^\omega$ and $J\subset\{1,\dots,k\}$ such that $\sum_{j\in J}p_j\geq c$, $F_j^L\subset F_j^M$ for all $j\in J$, and $L\cap M\subset F_L\cap F_M$.*

Remark. The Matching Lemma implies that the above sufficient condition is satisfied with $c=\frac{1}{2}$ provided that we also require $F_1^M<\ldots<F_k^M$ for all $M\in[\mathbb{N}]^\omega$.

Proof. We will verify that the stated condition implies the sufficient and necessary condition of Proposition 8.1. Given $\delta\in(0,1]$, assume that we are given a normalized, weakly null sequence (x_i), a continuous selection $M\mapsto(\boldsymbol{a}_M,x_M^*,F_M)$ of witnesses so that (61)–(65) and (69) hold. After passing to a subsequence, if necessary, we may assume that $\epsilon<c/48$ and all of the conditions (61)–(70) hold. Let $w=\sum_{j=1}^k w_j$, and set $p_j=w_j/w$ for each $j=1,\dots,k$. By our assumption we can find $L,M\in[\mathbb{N}]^\omega$ and $J\subset\{1,\dots,k\}$ such that $\sum_{j\in J}p_j\geq c$, $F_j^L\subset F_j^M$ for all $j\in J$, and $L\cap M\subset F_L\cap F_M$. Note that $\sum_{j\in J}w_j\geq cw\geq cK'(\delta)/4-\epsilon$. We now obtain a sequence of inequalities in a way very similar to that at the end of the proof of Theorem 2.1.

$$\begin{aligned} x_L^*(x_M) &\geq \sum_{i\in L\cap M} a_i^M x_L^*(x_i)-2\epsilon \\ &\geq \frac{1}{2}\sum_{j=1}^k\sum_{i\in F_j^L\cap F_j^M} a_i^L x_L^*(x_i)-2\epsilon \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}\sum_{j\in J} w_j - 2\epsilon \\ &\geq \frac{cK'(\delta)}{8} - 3\epsilon \geq \frac{c}{16}K'(\delta). \end{aligned}$$

□

The discrete nature of the sufficient condition of Proposition 8.2 makes it very attractive: it reduces Problem 1.2 to a combinatorial, Ramsey type problem. The conclusion in this condition is about "matching" the part of the combinatorial data of L and M that comes from the discretization of the coefficients of x_L and x_M, and it "ignores" the dual coefficients. We will now study the entire combinatorial data as an abstract object (i.e. we forget about the underlying Banach space). This leads to the introduction of resolutions. We will use them to discuss the possibility of a negative answer to Problem 1.2. To conclude this section we shall produce an example to show that $\sup_{\delta>0} K'(\delta)$ is strictly greater than 1 (recall that if (x_i) is a normalized, weakly null sequence with spreading model not equivalent to the unit vector basis of c_0, then for any $\epsilon > 0$ there is a subsequence (y_i) of (x_i) such that $K'((y_i), \delta) < 1+\epsilon$).

Let $k \in \mathbb{N}$. A *k-pattern* is a finite sequence in the set $\{1, \ldots, k\}$ (the numbers $1, \ldots, k$ will be called *colours*). A *k-resolution* is a pair $r = ((c_i)_{i=1}^n, (\alpha_i)_{i=1}^n)$, where $(c_i)_{i=1}^n$ is a k-pattern, and $(\alpha_i)_{i=1}^n$ are positive, real numbers. When we work with a fixed k we shall simply say pattern and resolution, respectively.

Let r be a k-resolution. The *weight of colour j in r* is

$$w_j(r) = \sum_{i\colon c_i=j} \alpha_i, \qquad j = 1, \ldots, k,$$

and the *weight of r* is $w(r) = \sum_{j=1}^k w_j(r)$. A pair (x, x^*) of elements of c_{00} *has resolution r* (or (x, x^*) is a *representation of r*) if the non-zero co-ordinates of x are $(2^{-c_i})_{i=1}^n$ in this order, and the non-zero co-ordinates of x^* are $(2^{c_i}\alpha_i)_{i=1}^n$ in this order, and moreover x and x^* have the same support. In other words, we have $x = \sum_{i=1}^n 2^{-c_i} e_{l_i}$ and $x^* = \sum_{i=1}^n 2^{c_i}\alpha_i e_{l_i}$ for some $1 \leq l_1 < \ldots < l_n$. Note that $x^*(x) = \sum_{i=1}^n \alpha_i = w(r)$.

Given $k \in \mathbb{N}$ and non-negative, real numbers $w_1, \ldots, w_k$ (called *weights*) with $\sum_{j=1}^k w_j = 1$, we let $\mathcal{R} = \mathcal{R}(w_1, \ldots, w_k)$ be the class of all k-resolutions r with $w_j(r) = w_j$ for each $j = 1, \ldots, k$. The necessary and sufficient condition of Proposition 8.1 motivates the following definition. Given $r, s \in \mathcal{R}$ we let

$$[r, s] = \max x^*(y),$$

where the maximum is over all pairs (x,x^*) and (y,y^*) of elements of c_{00} that have resolutions r and s, respectively. We also let $\langle r,s\rangle = \max\big\{[r,s],[s,r]\big\}$. Note that $[r,s]\leq \sum_{j=1}^k 2^{j-1}w_j$ for all $r,s\in\mathcal{R}$.

Given k-patterns $c=(c_i)_{i=1}^m$ and $d=(d_i)_{i=1}^n$, we write $c\subset d$ if there exist $1\leq l_1<\ldots<l_m\leq n$ such that $c_i=d_{l_i}$ for $i=1,\ldots,m$. Observe that if $r=(c,\alpha)$ and $s=(d,\beta)$ are elements of $\mathcal{R}$ and $c\subset d$, then $[r,s]\geq 1$. More generally, if we can find representations (x,x^*) and (y,y^*) of r and s, respectively, and a set $J\subset\{1,\ldots,k\}$ so that $\{i\in\mathbb{N}:\ x_i=2^{-j}\}\subset\{i\in\mathbb{N}:\ y_i=2^{-j}\}$ for each $j\in J$, then we have $[r,s]\geq x^*(y)\geq\sum_{j\in J}w_j$ (this observation is motivated by Proposition 8.2). Since for any $j\in\{1,\ldots,k\}$ we can find representations (x,x^*) and (y,y^*) such that the sets $\{i\in\mathbb{N}:\ x_i=2^{-j}\}$ and $\{i\in\mathbb{N}:\ y_i=2^{-j}\}$ are comparable, we have $\langle r,s\rangle\geq\max w_j\geq 1/k$ for all $r,s\in\mathcal{R}$.

Given $r,s\in\mathcal{R}$ and $\eta\in(0,1)$, we say that r and s are η-*orthogonal*, in symbols $r\perp_\eta s$, if $\langle r,s\rangle<\eta$. Note that this can only happen for $\eta>1/k$. Roughly speaking, if one could find for each $k\in\mathbb{N}$ an infinite set of pairwise $\eta(k)$-orthogonal resolutions with $\eta(k)\to 0$ as $k\to\infty$, then one could 'code' an example in a way reminiscent of the Maurey-Rosenthal construction [18] to show that $\sup_{\delta>0}L(\delta)=\infty$, where L is the function given in Definition 4.1. We sketch this next.

Example 8.3. Let $k\in\mathbb{N}$, $\eta=\eta(k)\in(0,1)$ and $C=C(k)\geq 1$. Assume that we can find weights $w_1,\ldots,w_k$ and a sequence (r_i) in $\mathcal{R}=\mathcal{R}(w_1,\ldots,w_k)$ so that $\langle r_i,r_j\rangle<\eta$ whenever $i\neq j$, and $\langle r_i,r_i\rangle\leq C$ for all $i\in\mathbb{N}$. Assume also that if $r_i=(c^{(i)},\alpha^{(i)})$, then $\max_j 2^{c_j^{(i)}}\alpha_j^{(i)}\leq 1$ for all $i\in\mathbb{N}$, and $\max_j 2^{c_j^{(i)}}\alpha_j^{(i)}\to 0$ as $i\to\infty$. (note that this is not a serious assumption: the resolutions in a large family of pairwise orthogonal elements of $\mathcal{R}$ are necessarily "flat" — c.f. proof of Proposition 8.6). We will now show that $L(2^{-k})\geq 1/(2C+6)\eta$. In particular, if $(C(k))_{k=1}^\infty$ is bounded and $\eta(k)\to 0$ as $k\to\infty$, then this solves Problem 1.2 in the negative.

Let Q be the set of all representations of the resolutions $r_i,\ i\in\mathbb{N}$. Let us fix an injective function ϕ (the coding function) that maps finite sequences of elements of Q to positive integers. A sequence $(x_j,x_j^*)_{j=1}^k$ of pairs of elements of c_{00} is called a *special sequence* if there exist positive integers l_j for $j=1,\ldots,k$ such that the following hold.

(72) $x_1^*<\ldots<x_k^*$,

(73) (x_j,x_j^*) has resolution r_{l_j} for $j=1,\ldots,k$,

(74) $l_j=\phi\big((x_1,x_1^*),\ldots,(x_{j-1},x_{j-1}^*)\big)$ for $j=1,\ldots,k$.

We then call the sum $\sum_{j=1}^k x_j^*$ a *special functional.* Let $\mathcal{F}$ be the set of all special functionals, and let us define a norm on c_{00} by letting

$$\|x\| = \|x\|_{\ell_\infty} \vee \sup\big\{|x^*(Ex)| : x^* \in \mathcal{F},\ E \in \mathcal{I}\big\}.$$

Here $\mathcal{I}$ denotes the set of intervals of positive integers, and Ex is the projection of x onto E. Let X be the completion of c_{00} in this norm. Then (e_i) is a normalized, bimonotone, weakly null basis of X. Let $M \in [\mathbb{N}]^\omega$. One can clearly choose a special sequence $(x_j, x_j^*)_{j=1}^k$ such that $\operatorname{supp}(x_j) \subset M$ for each $j = 1, \dots, k$. Using the injectivity of ϕ and the orthogonality of the resolutions r_i, it is not difficult to show that $\left\|\sum_{j=1}^k (-1)^j x_j\right\| \le 1 + C + 2k\eta \le (C+3)k\eta$, whereas

$$\Big\|\sum_{j\text{ odd}} x_j\Big\| + \Big\|\sum_{j\text{ even}} x_j\Big\| \ge \sum_{j=1}^k x_j^*\Big(\sum_{j=1}^k x_j\Big) = k.$$

This shows that $L((e_i)_{i\in M}, 2^{-k}) \ge 1/(2C+6)\eta$.

Our next result together with an earlier observation shows that a Maurey-Rosenthal-type example as described above is far from possible. Indeed, it shows that for all $k \in \mathbb{N}$ and for all weights $w_1, \dots, w_k$, any infinite subset $\mathcal{S}$ of $\mathcal{R}(w_1, \dots, w_k)$ contains a further infinite subset $\mathcal{S}'$ such that $\langle r, s\rangle \ge 1$ for all $r, s \in \mathcal{S}'$.

Proposition 8.4. *Let $k \in \mathbb{N}$. Given k-patterns $c^{(i)}$, $i \in \mathbb{N}$, there exist $1 \le l_1 < l_2 < \dots$ such that $c^{(l_i)} \subset c^{(l_{i+1})}$ for all $i \in \mathbb{N}$.*

Proof. We apply induction on k. When $k = 1$ the result is trivial. Now assume that $k > 1$. For each $i \in \mathbb{N}$ we can write

$$c^{(i)} = \big(c^{(i,1)}, c_1^{(i)}, c^{(i,2)}, c_2^{(i)} \dots, c^{(i,m_i)}, c_{m_i}^{(i)}, c^{(i,m_i+1)}\big),$$

where m_i is a non-negative integer, $c_j^{(i)}$ is a single colour (i.e. an element of $\{1, \dots, k\}$) and $c^{(i,j)}$ is a k-pattern using exactly the $k-1$ colours $\{1, \dots, k\} \setminus \{c_j^{(i)}\}$ for $1 \le j \le m_i$, and finally $c^{(i,m_i+1)}$ is a pattern (possibly of length zero) using strictly less than k colours. To see this simply trace the pattern $c^{(i)}$ from left to right and stop every time you have seen all k colours.

We consider two cases. In the first case $\sup_i m_i = \infty$. Let λ_i be the length of $c^{(i)}$ for each $i \in \mathbb{N}$. We can find $1 \le l_1 < l_2 < \dots$ such that $m_{l_{i+1}} > \lambda_{l_i}$ for all $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$ the pattern $c^{(l_{i+1})}$ is the concatenation of more than λ_{l_i} patterns each using all k colours, from which $c^{(l_i)} \subset c^{(l_{i+1})}$ is clear.

In the second case the sequence $(m_i)_{i=1}^\infty$ is bounded. Then after passing to a subsequence we may assume that for all $i \in \mathbb{N}$ we have $m_i = m$, $c_j^{(i)} = c_j$

for $1 \le j \le m$, and $c^{(i,m+1)}$ uses exactly the colours from a proper subset S of $\{1,\dots,k\}$. Then by the induction hypothesis we find $1 \le l_1 < l_2 < \dots$ such that $c^{(l_i,j)} \subset c^{(l_{i+1},j)}$ for all $i \in \mathbb{N}$ and for each $j = 1,\dots,m+1$. It follows that $c^{(l_i)} \subset c^{(l_{i+1})}$ for all $i \in \mathbb{N}$. □

Having seen that there are no pairwise η-orthogonal, infinite sets of resolutions for any $\eta \in (0,1]$, we now introduce so-called *Rademacher resolutions* that form arbitrarily large, finite sets of pairwise η-orthogonal resolutions for η of the order $1/\sqrt{k}$. This kills any hope of obtaining a positive answer to Problem 1.2 by proving a version of the Matching Lemma that allows us to find for any $N \in \mathbb{N}$, infinite sets $L_1,\dots,L_N$ whose combinatorial data match in a suitable way.

Let $k \ge 4$ and $k_0 = \lfloor \sqrt{k} \rfloor$. Set $w_{jk_0} = 1/k_0$ for each $j = 1,\dots,k_0$, and let $w_j = 0$ when j is not a multiple of k_0. We will now consider certain special elements of $\mathcal{R} = \mathcal{R}(w_1,\dots,w_k)$. Fix positive integers $n_1 < \dots < n_{k_0}$ satisfying

$$\sum_{1 \le j < j' \le k_0} \frac{n_j}{n_{j'}} < 2^{-k}. \tag{75}$$

For $n \in \mathbb{N}$ we denote by R_n the resolution (c,α), where

$$c = (k_0,\dots,k_0,2k_0,\dots,2k_0,\dots,k_0^2,\dots,k_0^2),$$

where colour jk_0 appears nn_j times, and $\alpha_i = 1/nn_jk_0$ whenever $c_i = jk_0$ (i.e. we distribute each weight uniformly over the corresponding colour). We will use the following notation: given $m \in \mathbb{N}$ and a resolution $r = (c,\alpha)$ we write $(r,\dots,r)_m$ for the resolution $s = (d,\beta)$, where $d = (c,\dots,c)$ with c repeated m times, and $\beta = (\alpha/m,\dots,\alpha/m)$ with α/m also repeated m times. Note that if r belongs to $\mathcal{R}$, then so does $(r,\dots,r)_m$ (indeed, this is true for any choice of weights $w_1,\dots,w_k$). Now given $l,n \in \mathbb{N}$, we define the *Rademacher* $R_{n,l}$ to be the resolution $(R_n,\dots,R_n)_{k_0^{l-1}}$. Note that $R_{n,l} \in \mathcal{R}$ for all $l,n \in \mathbb{N}$.

Proposition 8.5. *For all $m,n \in \mathbb{N}$, the Rademachers $R_{nk_0^{m-l},l}$, $l = 1,\dots,m$, are pairwise $5/k_0$-orthogonal. Moreover, $\langle R_{nk_0^{m-l},l}, R_{nk_0^{m-l},l}\rangle \le 1 + \frac{2}{k_0}$ for each $l = 1,\dots,m$.*

Proof. Fix $l,l' \in \{1,\dots,m\}$, let $r = R_{nk_0^{m-l},l} = (c,\alpha)$ and $s = R_{nk_0^{m-l'},l'} = (d,\beta)$. Choose representatives (x,x^*) and (y,y^*) of r and s, respectively, so that

$$[r,s] = x^*(y) = \sum_{i \in \mathrm{supp}(x) \cap \mathrm{supp}(y)} x_i^* y_i.$$

Note that each term $x_i^* y_i$ is equal to $2^{c_u}\alpha_u 2^{-d_v}$ for some u and v. Let S_1 (respectively, S_2 and S_3) be the set of all $i \in \mathbb{N}$ for which $c_u < d_v$ (respectively, $c_u > d_v$

and $c_u = d_v$). It is clear that

$$\sum_{i \in S_1} x_i^* y_i \leq 2^{-k_0} \sum_u \alpha_u = 2^{-k_0}.$$

For each $i \in S_2$ there exist $1 \leq j < j' \leq k_0$ such that $x_i^* = 2^{c_u} \alpha_u$ and $y_i = 2^{-d_v}$, where $c_u = j' k_0$, $\alpha_u = 1/nn_{j'} k_0^m$ and $d_v = jk_0$. Moreover, colour jk_0 occurs $nn_j k_0^{m-1}$ times in s. It follows that

$$\sum_{i \in S_2} x_i^* y_i \leq \sum_{1 \leq j < j' \leq k_0} 2^{(j'-j)k_0} \frac{1}{nn_{j'} k_0^m} nn_j k_0^{m-1} < \frac{1}{k_0},$$

by the choice of $n_1, \ldots, n_{k_0}$ (75). So the only significant contribution to $[r, s]$ comes from the set S_3 of co-ordinates, i.e. where the colours match. Here we always have the trivial estimate

$$\sum_{i \in S_3} x_i^* y_i \leq \sum_u \alpha_u = w(r) = 1.$$

In particular, when $l = l'$ this gives $\langle r, s \rangle \leq 1 + \frac{2}{k_0}$, as required. Note that for $i \in S_3$ we have $x_i = y_i = 2^{-jk_0}$ and $x_i^* = y_i^* = 2^{jk_0}/nn_j k_0^m$ for some $j \in \{1, \ldots, k_0\}$. In particular $x_i^* y_i = y_i^* x_i$, so when $l \neq l'$ we may without loss of generality assume that $l < l'$. Recall that for each $j \in \{1, \ldots, k_0\}$ colour jk_0 in r comes in k_0^{l-1} blocks, each block having length $nn_j k_0^{m-l}$. Consider such a block B, and suppose that a δ-proportion of the block corresponds to co-ordinates i of x that belong to S_3. The corresponding co-ordinates of y in turn correspond to colour-jk_0 bits of s. Since s is made up of $k_0^{l'-1}$ copies of $R_{nk_0^{m-l'}}$ and since colour jk_0 appears $nn_j k_0^{m-l'}$ times in each copy, the number of copies used up in this matching is at least

$$\frac{\delta nn_j k_0^{m-l}}{nn_j k_0^{m-l'}} = \delta k_0^{l'-l}, \tag{76}$$

which is strictly greater than 1 if $\delta > 1/k_0$. Also the contribution of a δ-proportion of block B to $\sum_{i \in S_3} x_i^* y_i$ is

$$\delta nn_j k_0^{m-l} \frac{1}{nn_j k_0^m} = \delta k_0^{-l}. \tag{77}$$

Let Δ be the sum of the δ's that are greater than $1/k_0$ over all colour-jk_0 blocks B of r and over all $j \in \{1, \ldots, k_0\}$. It follows from (76) that $\Delta k_0^{l'-l}$ is at most $2k_0^{l'-1}$, since the number of copies of $R_{nk_0^{m-l'}}$ that make up s is $k_0^{l'-1}$ and each copy is counted at most twice. Hence from (77) we obtain the estimate

$$\sum_{i\in S_3} x_i^* y_i \le \Delta k_0^{-l} + \frac{1}{k_0} w(r) \le \frac{3}{k_0}.$$

This finally shows that $[r,s]\le 5/k_0$, as required. □

Remarks. 1. We observed earlier that for any $r,s\in\mathcal{R}$ we have $[r,s]\ge\max w_j$, which in the above situation is $1/k_0$. Moreover, we always have $\langle r,r\rangle\ge 1$ for all $r\in\mathcal{R}$. So the measure of orthogonality we achieve is essentially best possible.
2. In Example 8.3 we required the resolutions in the pairwise orthogonal family to be 'flat'. Note that this holds for the Rademachers. Given $m,n\in\mathbb{N}$ and $l\in\{1,\dots,m\}$, if $R_{nk_0^{m-l},l}=(c,\alpha)$, then $\max_i 2^{c_i}\alpha_i\le 2^k/nn_1k_0^m\to 0$ as $m\to\infty$.

It is possible to measure, for each $\eta\in(0,1)$, the complexity of the family of finite sets of pairwise η-orthogonal resolutions by introducing a suitable ordinal index. We shall not do that, but simply comment that the above result would then say that for $\eta>5/\sqrt{k}$ and under the assumption that we only use colours that are multiples of $\sqrt{k}$ and carry equal weights, this complexity is at least ω. Whereas our next result shows that the complexity never exceeds ω (and this holds for general weights). So in some sense the set of resolutions has just enough complexity to allow the possibility of a negative asnwer to Problem 1.2.

Proposition 8.6. *Assume that $k\in\mathbb{N}$ and $w_1,\dots,w_k$ are arbitrary weights. Let $\mathcal{R}=\mathcal{R}(w_1,\dots,w_k)$ and $\eta\in(0,1/4)$. For all $r\in\mathcal{R}$ there exists $n\in\mathbb{N}$ so that whenever $s_1,\dots,s_n\in\mathcal{R}$ are pairwise η-orthogonal, we have $[r,s_i]\ge 1/2$ for each $i=1,\dots,n$.*

Proof. Choose j_0 and j_1 minimal so that

$$\sum_{j=1}^{j_0} w_j \ge 1/4 \qquad \textit{and} \qquad \sum_{j=1}^{j_1} w_j \ge 1/2.$$

We then have

$$\sum_{j=j_0}^{j_1} w_j \ge 1/4 \qquad \textit{and} \qquad \sum_{j=j_1}^{k} w_j \ge 1/2.$$

Now assume the result is false. Then there exists $r\in\mathcal{R}$ such that for all $n\in\mathbb{N}$ we have $\mathcal{R}_n\subset\mathcal{R}$ and $t_n\in\mathcal{R}_n$ such that $|\mathcal{R}_n|\ge n$, $\langle t,t'\rangle<\eta$ for all $t,t'\in\mathcal{R}_n$ with $t\ne t'$ and $[r,t_n]<1/2$. We now verify two claims.

First observe that for each $n\in\mathbb{N}$ the number of co-ordinates of t_n of colours $1,\dots,j_1$ is at most the length of r. Indeed, otherwise we can choose representatives (x,x^*) of r and (y,y^*) of t_n so that whenever $x_i=2^{-j}$ for some $j\geq j_1$, then $y_i=2^{-j'}$ for some $j'\leq j_1$, and this would give $[r,t_n]\geq w_{j_1}+\ldots+w_k\geq 1/2$.

Secondly, we claim that for all $n\in\mathbb{N}$ and for all $t\in\mathcal{R}_n$ the number of co-ordinates of t of colours $1,\dots,j_0$ is at most the length of r. Otherwise by the first claim we can find representatives (x,x^*) of t_n and (y,y^*) of t so that whenever $x_i=2^{-j}$ for some $j_0\leq j\leq j_1$, then $y_i=2^{-j'}$ for some $j'\leq j_0$, and this would give $[t_n,t]\geq w_{j_0}+\ldots+w_{j_1}\geq 1/4$.

Now by simple pigeonhole principle, if n is greater than the number of patterns of length at most the length of r in colours $1,\dots,j_0$, then there exist distinct $t,t'\in\mathcal{R}_n$ so that the patterns in t and t' formed by the colours $1,\dots,j_0$ are identical. It follows that there exist representatives (x,x^*) of t and (y,y^*) of t' so that $\{i\in\mathbb{N}:\ x_i=2^{-j}\}=\{i\in\mathbb{N}:\ y_i=2^{-j}\}$ for each $j=1,\dots,j_0$, and hence we obtain the contradiction $\langle t,t'\rangle\geq w_1+\ldots+w_{j_0}\geq 1/4$. $\square$

We conclude by constructing a relatively simple example using Rademachers to show that $\sup_{\delta>0}K'(\delta)\geq 5/4$.

Example 8.7. Let $\epsilon\in(0,1)$. Fix positive integers $n_1<n_2$ and K such that

$$\frac{n_1}{2n_2}+2^{-K}<\epsilon \qquad \text{and} \qquad \frac{2n_1+n_2}{n_1 2^K}<1.$$

For an infinite subset $M=\{m_1<m_2<\ldots\}$ of $\mathbb{N}$ set $n_M=(n_1+n_2)2^{Km_2-1}$ and let

$$E_M=\{m_3,m_4,\dots,m_{n_M+2}\}.$$

Now write E_M as a union

$$E_M=\bigcup_{j=1}^{2^{Km_1-1}} I_j^M \ \cup \ \bigcup_{j=1}^{2^{Km_1-1}} J_j^M,$$

where $I_1^M<J_1^M<I_2^M<J_2^M<\ldots<I^M_{2^{Km_1-1}}<J^M_{2^{Km_1-1}}$ and $|I_j^M|=n_1 2^{Km_2-Km_1}$ and $|J_j^M|=n_2 2^{Km_2-Km_1}$ for each $j=1,\dots,2^{Km_1-1}$. Finally, set

$$E_1^M=\bigcup_{j=1}^{2^{Km_1-1}} I_j^M \qquad \text{and} \qquad E_2^M=\bigcup_{j=1}^{2^{Km_1-1}} J_j^M,$$

so we have $|E_1^M|=n_1 2^{Km_2-1}$ and $|E_2^M|=n_2 2^{Km_2-1}$. Note that if we let $c_i=2$ whenever $m_{i+2}\in E_1^M$ and $c_i=4$ whenever $m_{i+2}\in E_2^M$, then (c_i) is the pattern of the Rademacher resolution $R_{2^{Km_2-Km_1},Km_1}$ as defined preceding Proposition 8.5 when $k=4$.

We shall denote by $\mathbb{I}_F$ the indicator function of a set $F\subset\mathbb{N}$, which is also the element $\sum_{i\in F} e_i$ of c_{00}. Given $M=\{m_1<m_2<\dots\}\in[\mathbb{N}]^\omega$, let

$$\begin{array}{rcl} x_M &=& -\tfrac{1}{2}e_{m_1} + \tfrac{1}{2}e_{m_2} + \tfrac{1}{2}\mathbb{I}_{E_1^M} + \tfrac{1}{4}\mathbb{I}_{E_2^M}, \\ x_M^+ &=& \tfrac{1}{2}e_{m_2} + \tfrac{1}{2}\mathbb{I}_{E_1^M} + \tfrac{1}{4}\mathbb{I}_{E_2^M}, \\ x_M^* &=& \tfrac{1}{2}e_{m_1} + e_{m_2} + \tfrac{1}{|E_1^M|}\mathbb{I}_{E_1^M} + \tfrac{1}{|E_2^M|}\mathbb{I}_{E_2^M}. \end{array}$$

Define a norm on c_{00} by setting

$$\|x\| = \|x\|_{\ell_\infty} \vee \sup\left\{|x_M^*(Ex)| :\ M\in[\mathbb{N}]^\omega,\ E\in\mathcal{I}\right\}$$

for each $x\in c_{00}$. Here $\mathcal{I}$ denotes the set of initial segments of $\mathbb{N}$. Let X be the completion of $(c_{00},\|\cdot\|)$. It is easy to verify that (e_i) is a normalized, weakly null, monotone basis of X. We are going to show that for any subsequence (f_i) of (e_i) we have $K((f_i),1/4)\geq 5/4(1+\epsilon)$. Since ϵ was arbitrary, this shows that $K(1/4)\geq 5/4$.

Fix $M=\{m_1<m_2<\dots\}\in[\mathbb{N}]^\omega$. On the one hand we have

$$\|x_M^+\| \geq x_M^*(x_M^+) = \frac{5}{4}.$$

On the other hand, we are going to show that $\|x_M\|\leq 1+\epsilon$. So let us fix $L=\{l_1<l_2<\dots\}\in[\mathbb{N}]^\omega$. We need to estimate $x_L^*(Ex_M)$ for any $E\in\mathcal{I}$. This is always at least $-\frac{1}{2}$. To get an upper bound, we may clearly assume that $\mathrm{supp}(x_M)\subset E$. We now split into four cases. The first three of these use only the trivial estimate

$$x^*(y) = \sum_i x_i^* y_i \leq \min\left\{\|x^*\|_{\ell_\infty}\cdot\|y\|_{\ell_1},\ \|x^*\|_{\ell_1}\cdot\|y\|_{\ell_\infty}\right\}$$

for any $x^*, y\in c_{00}$.

Case 1. If $l_1=m_1$ and $l_2=m_2$, then we have

$$\begin{aligned} x_L^*(x_M) &= -\frac{1}{4} + \frac{1}{2} + \frac{1}{|E_1^L|}\mathbb{I}_{E_1^L}\Big(\frac{1}{2}\mathbb{I}_{E_1^M} + \frac{1}{4}\mathbb{I}_{E_2^M}\Big) \\ &\quad + \frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big(\frac{1}{2}\mathbb{I}_{E_1^M} + \frac{1}{4}\mathbb{I}_{E_2^M}\Big) \\ &\leq -\frac{1}{4} + \frac{1}{2} + 1\cdot\frac{1}{2} + \frac{1}{n_2 2^{Km_2-1}}\Big(\frac{1}{2}n_1 2^{Km_2-1} + \frac{1}{4}n_2 2^{Km_2-1}\Big) \\ &= 1 + \frac{n_1}{2n_2} < 1+\epsilon. \end{aligned}$$

Case 2. If $l_2<m_2$, then we have

$$\begin{aligned} x_L^*(x_M) &= x_L^*\Big(-\frac{1}{2}e_{m_1}\Big) \\ &\quad +\Big(\frac{1}{|E_1^L|}\mathbb{I}_{E_1^L}+\frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big)\Big(\frac{1}{2}e_{m_2}+\frac{1}{2}\mathbb{I}_{E_1^M}+\frac{1}{4}\mathbb{I}_{E_2^M}\Big) \\ &\leq 0+2\cdot\frac{1}{2}=1. \end{aligned}$$

Case 3. If $l_2>m_2$, then we have

$$\begin{aligned} x_L^*(x_M) &= \Big(\frac{1}{2}e_{l_1}+e_{l_2}\Big)(x_M) \\ &\quad +\Big(\frac{1}{|E_1^L|}\mathbb{I}_{E_1^L}+\frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big)\Big(\frac{1}{2}\mathbb{I}_{E_1^M}+\frac{1}{4}\mathbb{I}_{E_2^M}\Big) \\ &\leq \frac{3}{2}\cdot\frac{1}{2}+\frac{1}{n_1 2^{Kl_2-1}}\Big(\frac{1}{2}n_1 2^{Km_2-1}+\frac{1}{4}n_2 2^{Km_2-1}\Big) \\ &= \frac{3}{4}+\frac{2n_1+n_2}{4n_1 2^K}\leq 1. \end{aligned}$$

Case 4. If $l_2=m_2$ and $l_1\neq m_1$, then we have to use the structure of the Rademacher patterns to get an upper bound. The argument is along similar lines to the proof of Proposition 8.5. First we have

$$x_L^*(x_M)=\frac{1}{2}+\Big(\frac{1}{|E_1^L|}\mathbb{I}_{E_1^L}+\frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big)\Big(\frac{1}{2}\mathbb{I}_{E_1^M}+\frac{1}{4}\mathbb{I}_{E_2^M}\Big), \tag{78}$$

and

$$\frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big(\frac{1}{2}\mathbb{I}_{E_1^M}\Big)\leq\frac{1}{n_2 2^{Kl_2-1}}\cdot\frac{1}{2}\cdot n_1 2^{Km_2-1}=\frac{n_1}{2n_2}. \tag{79}$$

Also, since $|E_1^L\cap E_2^M|\leq|E_1^L|-|E_1^L\cap E_1^M|$, we have

$$\begin{aligned} \mathbb{I}_{E_1^L}\Big(\frac{1}{2}\mathbb{I}_{E_1^M}+\frac{1}{4}\mathbb{I}_{E_2^M}\Big) &= \frac{1}{2}|E_1^L\cap E_1^M|+\frac{1}{4}|E_1^L\cap E_2^M| \\ &\leq \frac{1}{4}|E_1^L\cap E_1^M|+\frac{1}{4}|E_1^L|. \end{aligned} \tag{80}$$

Let us now assume that $l_1<m_1$. For each $j=1,\dots,2^{Kl_1-1}$ set

$$A_j=\big\{i\in\{1,\dots,2^{Km_1-1}\}:\ I_i^M\cap I_j^L\neq\emptyset\big\}.$$

We now have

$$|E_1^L \cap E_1^M| = \sum_{j=1}^{2^{Kl_1-1}} \sum_{i\in A_j} |I_j^L \cap I_i^M| \leq \sum_{j=1}^{2^{Kl_1-1}} |A_j| n_1 2^{Km_2-Km_1}.$$

Hence from (80) we obtain

$$\frac{1}{|E_1^L|}\mathbb{I}_{E_1^L}\Big(\frac{1}{2}\mathbb{I}_{E_1^M} + \frac{1}{4}\mathbb{I}_{E_2^M}\Big) \leq \frac{1}{2}\cdot 2^{-Km_1}\sum_{j=1}^{2^{Kl_1-1}} |A_j| + \frac{1}{4}. \tag{81}$$

Since $E_2^L \cap J_i^M = \emptyset$ whenever $\min A_j \leq i < \max A_j$ for some $j \in \{1,\dots,2^{Kl_1-1}\}$, we have

$$|E_2^L \cap E_2^M| = \sum_{i=1}^{2^{Km_1-1}} |E_2^L \cap J_i^M| \leq \Big(2^{Km_1-1} - \sum_{j=1}^{2^{Kl_1-1}} (|A_j|-1)\Big) n_2 2^{Km_2-Km_1}.$$

It follows that

$$\frac{1}{|E_2^L|}\mathbb{I}_{E_2^L}\Big(\frac{1}{4}\mathbb{I}_{E_2^M}\Big) \leq \frac{1}{4} - \frac{1}{2}\cdot 2^{-Km_1}\sum_{j=1}^{2^{Kl_1-1}} |A_j| + 2^{Kl_1-Km_1}. \tag{82}$$

Note that $2^{Kl_1-Km_1} \leq 2^{-K}$ since we are assuming that $l_1 < m_1$. Putting together (78), (79), (81) and (82) we finally obtain

$$x_L^*(x_M) \leq 1 + \frac{n_1}{2n_2} + 2^{-K} < 1 + \epsilon, \tag{83}$$

as required. The case when $l_1 > m_1$ is very similar. For each $j = 1,\dots,2^{Km_1-1}$ set

$$A_j = \big\{ i \in \{1,\dots,2^{Kl_1-1}\} : \, I_i^L \cap I_j^M \neq \emptyset \big\}.$$

We then proceed as before making the obvious changes in the various summations.

Remarks. 1. Since $\|x_M^*\|_{\ell_1} \leq \frac{7}{2}$ for all $M \in [\mathbb{N}]^\omega$, the basis (e_i) of X is 7/2-equivalent to the unit vector basis of c_0, yet no subsequence is C-unconditional for $C < 5/4(1+\epsilon)$. So the above example also shows that $C(\delta) \geq 5/4$ whenever $\delta \leq 2/7$, where $C(\delta)$ is the constant introduced in Section 5 in relation to the c_0-problem.

2. The basis (e_i) of the space X constructed above is also an example of a normalized, weakly null sequence that has no quasi-greedy basic subsequence with constant strictly less than 8/7. To see this let $\alpha = 2/3$ and let

$$\begin{aligned} y_M &= -\alpha e_{m_1} + e_{m_2} + \mathbb{I}_{E_1^M} + \tfrac{2}{3}\mathbb{I}_{E_2^M}, \\ y_M^+ &= e_{m_2} + \mathbb{I}_{E_1^M} + \tfrac{2}{3}\mathbb{I}_{E_2^M} \end{aligned}$$

for each $M \in [\mathbb{N}]^\omega$ (following the notation in the proof above). Given $\epsilon > 0$ we may choose the parameters n_1, n_2 and K so that

$$\frac{\|y_M^+\|}{\|y_M\|} > \frac{8}{7} - \epsilon \tag{84}$$

for all $M \in [\mathbb{N}]^\omega$. This is proved by exactly the same calculation as in the proof above.

Now if $\alpha = 2/3 - \eta$ for some $\eta > 0$, then (84) still holds provided η is sufficiently small. Then y_M^+ is the projection of y_M onto the set of co-ordinates where the size of the coefficient is at least 2/3. It follows that $(e_i)_{i \in M}$ is not quasi-greedy with constant $8/7 - \epsilon$ for any $M \in [\mathbb{N}]^\omega$.

References

[1] Alspach, Dale E.; Argyros, Spiros *Complexity of weakly null sequences.* Dissertationes Math. (Rozprawy Mat.) **321** (1992).

[2] Alspach, D.; Odell, E. *Averaging weakly null sequences.* Functional analysis (Austin, TX, 1986–87), 126–144, Lecture Notes in Math., 1332, Springer, Berlin, 1988.

[3] Argyros, S. A.; Gasparis, I. *Unconditional structures of weakly null sequences.* (English. English summary) Trans. Amer. Math. Soc. **353** (2001), no. 5, 2019–2058 (electronic).

[4] Argyros, Spiros A.; Godefroy, Gilles; Rosenthal, Haskell P. *Descriptive Set Theory and Banach Spaces.* Handbook of the Geometry of Banach Spaces, Vol. 2, 1007–1069, North-Holland, Amsterdam, 2003.

[5] Argyros, S. A., Mercourakis, S., Tsarpalias, A. *Convex unconditionality and summability of weakly null sequences.* Israel J. Math. **107** (1998), 157–193.

[6] Arvanitakis, Alexander D. *Weakly null sequences with an unconditional subsequence.* Proc. Amer. Math. Soc. **134** (2006), 67–74.

[7] Bollobás, Béla *Modern graph theory.* Graduate Texts in Mathematics, 184. Springer-Verlag, New York, 1998.

[8] Ellentuck, Erik *A new proof that analytic sets are Ramsey.* J. Symbolic Logic **39** (1974), 163–165.

[9] Elton, J. *Weakly null normalized sequences in Banach spaces.* Ph.D. thesis, Yale Univ., 1978.

[10] Dilworth, S. J.; Kalton, N. J.; Kutzarova, Denka *On the existence of almost greedy bases in Banach spaces. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday.* Studia Math. **159** (2003), no. 1, 67–101.

[11] Galvin, Fred; Prikry, Karel *Borel sets and Ramsey's theorem.* J. Symbolic Logic **38** (1973), 193–198.

[12] Gasparis, I.; Odell E.; Wahl, B. *Weakly null sequences in the Banach space $C(K)$.* preprint.

[13] Graham, Ronald L.; Rothschild, Bruce L.; Spencer, Joel H. *Ramsey theory.* Second edition. Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

[14] Gowers, W. T.; Maurey, B. *The unconditional basic sequence problem.* J. Amer. Math. Soc. **6** (1993), no.4, 851 – 874.

[15] James, Robert C. *Uniformly non-square Banach spaces.* Ann. of Math. (2) **80** 1964 542–550.

[16] Konyagin, S. V.; Temlyakov, V. N. *A remark on greedy approximation in Banach spaces.* East J. Approx. **5** (1999), no. 3, 365–379.

[17] Lopez-Abad, J.; Todorcevic, S. *Pre-compact families of finite sets of integers and weakly null sequences in Banach spaces.* preprint.

[18] Maurey, B.; Rosenthal, H. P. *Normalized weakly null sequence with no unconditional subsequence.* Studia Math. **61** (1977), no. 1, 77–98.

[19] Nash-Williams, C. St. J. A. *On well-quasi-ordering transfinite sequences.* Proc. Cambridge Philos. Soc. **61** 1965 33–39.

[20] Odell, E. *Applications of Ramsey theorems to Banach space theory.* Notes in Banach spaces, pp. 379–404, Univ. Texas Press, Austin, Tex., 1980.

[21] Odell, E. *On Schreier unconditional sequences.* Banach spaces (Mérida, 1992), 197–201, Contemp. Math., **144**, Amer. Math. Soc., Providence, RI, 1993.

[22] Odell, E.; Schlumprecht, Th. *The distortion of Hilbert space.* Geom. Funct. Anal. **3** (1993), no. 2, 201–207.

[23] Pudlák, Pavel; Rödl, Vojtěch, *Partition theorems for systems of finite subsets of integers.* Discrete Math. **39** (1982), no. 1, 67–73.

[24] Rosenthal, Haskell P. *A characterization of Banach spaces containing l^1.* Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

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