Why do we have to learn proofs!?

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That's right. You are going to have to endure proofs. Like many of my students, perhaps you are asking yourself (or me), why do I have to learn proofs? Aren't they just some esoteric, jargon-filled, technical writing that only a professional mathematician would care about?

Well, no. And I'd like to offer a short justification of this claim. My argument is three-fold: (1) proofs are all around you, (2) it's quite possible to get better at them by practice and by benefiting from the accumulated knowledge of two thousand years of mathematicians, and (3) this will really help you in "real life," whether you go into mathematics, carpentry, or child-rearing.

First and foremost, it is important to know what a proof is. It is *not* one of those horrible two-column tables of axioms and deductions you saw in 10th grade geometry. That is a bizarre (though sometimes useful) invention of mathematics educators which constitutes a particular way to write down a very special kind of proof in a very narrow area. No mathematician would be caught dead writing such a thing[1]. A proof is *not* some long sequence of equations on a chalk board, nor is it a journal article. These things are ways that mathematician communicate proofs, but the truth is, *proof is in your head.*

A proof is an argument, a justification, a reason that something is true. It's got to be a particular kind of reasoning – logical – to be called a proof. (There are certainly plenty of other, equally valid forms of reasoning. And some of them are even used in "doing" mathematics. But they're not proofs.) A proof is just the answer to the question "Why?", when the person asking the question wants an argument that is indisputable, in the sense that any person of normal intelligence who has enough time could be convinced of it. "Why is the sky blue?" is a question any answer to which could conceivably be wrong: perhaps we will all wake up tomorrow, and find it orange. It's possible, both physically and metaphysically. (In fact, in New York City, this happens quite frequently.) On the other hand, if one asks, "Why does $2+2 = 2 \times 2$," or, "Why is the area of a rectangle the product of its side lengths?", I can give you an answer that's impossible to argue with. If we agree on the premises – the definitions of the terminology being used – and we agree that, if A implies B, and A is true, then B is true, then we agree on the conclusion. There's no way around it.[2]

In fact, you use "proofs" all the time. When you reason with yourself that it will be cheaper to buy the larger cans of beans, you are proving something about the respective prices. When you play that terribly addictive game Minesweeper, and you see that, oh, yeah, there *has* to be a mine under *this* block, well, you've proven something to yourself. When you up the ante because you've got three kings, and you know who's got the aces, you're proving something about the possible hands that your

[1] Logicians excepted.

[2] W. V. O. Quine is rolling in his grave.

friends hold. When you know you can't turn right on red in at a certain intersection, you've constructed a proof of the illegality of this move. (It goes something like this: "Normally, I have to stop at a red light. But I'm in the right lane, so that rule is superceded by the state law that right-on-red is okay. However, there's a sign right there that says it's not okay at this intersection. So it wouldn't be legal to turn."[3]) Many, many real-life situations require reasoning that is, at least in part, something that would qualify as a mathematical proof. At the very least, mathematical-type reasoning is a powerful addition to anyone's critical thinking toolbox, applicable in a wide variety of settings. Mathematics is also a cornerstone of the sciences, which in turn provide a profoundly useful way to understand and interpret the world around us.

Okay, so what? Why are you, a Drama major, being subjected to the requirement that you write down a so-called "proof" in some crazy mathematical hieroglyphics about constructs like integrals and binomial coefficients that you'll never see again? The reason is that mathematicians have spent more than two millenia working out what proofs are and distilling them out of the complex and unruly real world. The idea is that studying the *concept* of proof will make you better at it in all sorts of situations. We *could* muck it up by giving you problems involving other types of reasoning, too, by why not just boil it down to the mathematics itself? (Word problems are sort of a middle-ground: they invite you to figure out what sorts of argument you're going to need, and you're supposed to produce a proof and then apply it to the ridiculously idealized circumstances described.) And the reason that the proofs you are dealing with involve mathematical constructs is that mathematical constructs are things that have precise definitions one can reason about. Notwithstanding what your mathematics-obsessed philosophy professor told you[4], you can't prove anything about thoughts, lightning, or yodelling. They're literally impossible to encapsulate in mathematical language, though often it is possible to "model" them by idealizing the situation first.

Well, that's the reason that practicing proving things is a good idea, but why must you write it down? No, it's not because I'm sadistic and proofs are my instrument of torture. (Not to say that all math professors aren't sadistic, as this is decidedly not true.) Here are a few real reasons. First of all, because otherwise I can't tell if you know why something is true! You haven't learned anything if you get the final answer by luck (or by good peripheral vision). If I could read your mind, I could check if you thought through the problem "correctly," i.e., how clear your reasoning is. But I can't, so you would you please just write down your thinking so I can see if you get it? Second, the very act of writing it down serves as an extremely effective way to cut away the chaff, straighten out your thinking, and force you to work out the details – which often are at the heart of the problem – instead of just producing some sort of vague, hand-wavy, it-could-be-turned-into-a- proof. Third, a written representation of a proof, even if it's just a sequence of equations, is a way to check and see if everything is kosher with your argument. Finally, and this applies

[3]Thank Rob Hochberg for this one.

[4]See *proofs, inappropriate applications* and *fetishization, other disciplines*.

particularly to people in the sciences or engineering, you might have to write down proofs or pseudo-proofs in subsequent coursework or even (gasp!) real life.

Now, given that I've convinced you (haven't I?) that proofs are useful, let me assure you that you can do them. You do them all the time. I've literally never met a student who couldn't prove things, although I've met plenty of students unwilling to or lacking the time to learn how to. Perhaps you will need some practice when it comes to reasoning about such abstract objects that come up in a math class, but that's just a matter of doing your problem sets. Furthermore there are lots of proof "techniques" that can guide the way and make the work almost formulaic once you get the hang of it. (Induction comes to mind here.) And getting the hang of it means practice, as annoying as it is. Believe me, I sympathize with your urge to say, "I get it, why do I have to do it fifty times?" But the truth is, you don't know if you get it until you've done it at least a few times, and you're unlikely to retain it for very long unless you do it a few more. Very often you think you know what's going on, until a problem comes up with a twist in it, and you realize that you haven't quite grokked the big picture yet.

I hope that explains why you're being tormented so with proofs. Written proofs are a record of your understanding, and a way to communicate mathematical ideas with others. "Doing" mathematics is all about finding proofs. And real life has a lot to do with "doing" mathematics, even if it doesn't look that way very often.