MATROID THEORY NOTES

MATH 777, SPRING 2008, COOPER

1. Axiom Systems

The axioms for an independence system $(E, \mathcal{I})$:

(I1): $\emptyset \in \mathcal{I}$.
(I2): If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
(I3): If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then $\exists e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The axioms for a basis system $(E, \mathcal{B})$:

(B1): $\mathcal{B} \neq \emptyset$.
(B2): If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then $\exists y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

The axioms for a circuit system $(E, \mathcal{C})$:

(C1): $\emptyset \notin \mathcal{C}$.
(C2): If $C_1, C_2 \in \mathcal{C}$, and $C_1 \subseteq C_2$, then $C_1 = C_2$.
(C3): If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then $\exists C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

The axioms for a rank function $r : 2^E \to \mathbb{Z}^+$:

(R1): For $X \subseteq E$, $0 \leq r(X) \leq |X|$.
(R2): If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.
(R3): If $X, Y \subseteq E$, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

The axioms for a closure function $\overline{\cdot} : 2^E \to 2^E$:

(CL1): For $X \subseteq E$, $X \subseteq \overline{X}$.
(CL2): If $X \subseteq Y \subseteq E$, then $\overline{X} \subseteq \overline{Y}$.
(CL3): If $X \subseteq E$, then $\overline{X} = \overline{\overline{X}}$.
(CL4): If $X \subseteq E$, $x \in E$ and $y \in \overline{X \cup x} - \overline{X}$, then $x \in \overline{X \cup y}$.

The axioms for a system $\mathcal{F} \subset 2^E$ of flats:

(F1): $E \in \mathcal{F}$.
(F2): If $F_1, F_2 \subseteq \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.
(F3): If $F \in \mathcal{F}$ and $\{F_1, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ that properly contain F, then the sets $F_1 \setminus F, \ldots, F_k \setminus F$ partition $E \setminus F$.

The axioms for a spanning set system $\mathcal{S} \subset 2^E$:

(S1): $\mathcal{S} \neq \emptyset$.
(S2): If $S_1 \in \mathcal{S}$ and $S_2 \supseteq S_1$, then $S_2 \in \mathcal{S}$.
(S3): If $S_1, S_2 \in \mathcal{S}$ and $|S_1| > |S_2|$, then $\exists e \in S_1 - S_2$ such that $S_1 - e \in \mathcal{S}$.

2. Translations

The following set of "cryptomorphisms" between definitions of a matroid is strongly connected as a digraph, and therefore provides a complete (though not always maximally efficient) translation mechanism between any two.

$\mathcal{I} \to \mathcal{B}$: $\mathcal{B}$ is the set of maximal elements of $\mathcal{I}$.
$\mathcal{B} \to \mathcal{I}$: $\mathcal{I} = \{I : I \subseteq B, B \in \mathcal{B}\}$.
$\mathcal{I} \to \mathcal{C}$: $\mathcal{C}$ is the set of minimal elements of $2^E \setminus \mathcal{I}$.
$\mathcal{C} \to \mathcal{I}$: $\mathcal{I} = \{I : C \not\subseteq I, \forall C \in \mathcal{C}\}$.
$r \to \mathcal{I}$: $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$.
$\mathcal{I} \to r$: $r(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}\}$.
$r \to \overline{\cdot}$: $\overline{X} = \{x \in E : r(X \cup x) = r(X)\}$.
$\overline{\cdot} \to \mathcal{I}$: $\mathcal{I} = \{X \subseteq E : \forall x \in X, x \notin \overline{X - x}\}$.
$\overline{\cdot} \to \mathcal{F}$: $\mathcal{F} = \{\overline{X} : X \subseteq E\}$.
$\mathcal{F} \to \overline{\cdot}$: $\overline{X} = \bigcap\{F : F \in \mathcal{F}, F \supseteq X\}$.
$r \to \mathcal{S}$: $\mathcal{S} = \{S : r(S) = r(E)\}$.
$\mathcal{S} \to \mathcal{B}$: $\mathcal{B} = \{B \subseteq E : \forall X \subseteq E, B \cup X \in \mathcal{S}\}$.

3. Examples

Let $X \subseteq E$ be any subset throughout the following.

1. Graphic Matroids

E	$E(G)$
$\mathcal{I}$	acyclic sets
$\mathcal{B}$	spanning forests
$\mathcal{C}$	cycles
$r(X)$	$\lvert X\rvert -$ number of components of X as a subgraph
$\overline{X}$	$X \cup$ any $e \in E$ so that $X \cup e$ has more cycles than X

$\mathcal{F}$	no broken cycles, i.e., if C is a cycle of G, then $\|C \setminus X\| \neq 1$.
$\mathcal{S}$	spanning subgraphs

2. Linear Matroids, aka Vector Matroids

E	elements of a (finite) vector space V
$\mathcal{I}$	independent sets
$\mathcal{B}$	bases
$\mathcal{C}$	minimally dependent sets
$r(X)$	$\dim(\text{span}(X))$
$\overline{X}$	$\text{span}(X)$
$\mathcal{F}$	subspaces
$\mathcal{S}$	sets spanning V

3. Algebraic Matroids

E	finite subset of a field extension K/F
$\mathcal{I}$	algebraically independent sets (i.e., if $X = \{x_1, \ldots, x_n\}$, and $f(x_1, \ldots, x_n) = 0$ for $f \in F[t_1, \ldots, t_n]$, then $f \equiv 0$).
$\mathcal{B}$	transcendence bases for $F(E)$, i.e., minimal sets X so that $F(E)$ is algebraic over $F(X)$
$\mathcal{C}$	minimally dependent sets
$r(X)$	transcendence degree of $F(X)/F$
$\overline{X}$	$F(X)^{\text{alg}} \cap E$, where $\cdot^{\text{alg}}$ denotes algebraic closure
$\mathcal{F}$	$F(X)^{\text{alg}} \cap E$, for various $X \subseteq E$
$\mathcal{S}$	sets X so that $F(E)$ is algebraic over $F(X)$

4. Uniform Matroids

E	any finite set
$\mathcal{I}$	$\binom{E}{\leq k}$
$\mathcal{B}$	$\binom{E}{k}$
$\mathcal{C}$	$\binom{E}{k+1}$
$r(X)$	$\min\{\lvert X\rvert, k\}$
$\overline{X}$	X if $\lvert X\rvert < k$, E otherwise
$\mathcal{F}$	$\binom{E}{<k} \cup \{E\}$
$\mathcal{S}$	$\binom{E}{\geq k}$

5. Fano Plane

For ease of notation, we write $a_1 \cdots a_k$ for $\{a_1, \ldots, a_k\}$.

E	$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
$\mathcal{I}$	$\binom{E}{2} \cup (\mathbb{Z}_7 + \{012, 014, 015, 024, 0125\})$
$\mathcal{B}$	$\mathbb{Z}_7 + \{0125\}$
$\mathcal{C}$	$\{013, 124, 235, 346, 045, 156, 026\} = \mathbb{Z}_7 + \{013\}$
$r(X)$	no simple description
$\overline{X}$	no simple description
$\mathcal{F}$	no simple description
$\mathcal{S}$	no simple description