Translate the following (true) statements into the Predicate Calculus using the domain given and only the predicates given and algebraic expressions.

1. “An integer is prime if and only if its only divisors are 1 and itself.” Domain = \( \mathbb{Z} \), Predicates: \( P(n) \), meaning “\( n \) is prime”, and \( D(a,b) \), meaning “\( a \) divides \( b \)”. 

2. “The sum of two continuous functions is a continuous function.” Domain = functions (say, \( \mathbb{R} \rightarrow \mathbb{R} \)), Predicate: \( C(f) \), meaning “\( f \) is continuous”. 

3. “If \( x \) is a rational number, but not 0 or 1, and \( y = \sqrt{n} \) is irrational, then \( x^y \) is irrational.” Domain = \( \mathbb{R} \), Predicate: \( Q(t) \), meaning “\( t \) is rational”. 

4. “The only consecutive perfect powers are 0, 1 and 8, 9.” Domain = \( \mathbb{N} \). Recall that a natural number \( n \) is called a “perfect power” if it can be written \( n = a^b \) for some \( a \) and \( b \) natural numbers with \( b \geq 2 \). 

5. “If \( d \geq 3 \), then \( a^d + b^d = c^d \) has no solutions.” Domain = \( \mathbb{Z}^+ \) (the positive integers). 

6. “The number \( a^2 \) is either divisible by 4 or one more than a number divisible by 4.” Domain = \( \mathbb{Z} \), Predicate: \( D(a,b) \), meaning “\( a \) divides \( b \)”. 

7. “If \( ab \) is rational and nonzero, then either both \( a \) and \( b \) are rational or both \( a \) and \( b \) are irrational.” Domain = \( \mathbb{R} \), Predicate: \( Q(t) \), meaning “\( t \) is rational”. 

8. “Any function from a finite set to another finite set of the same cardinality which is one-to-one is also onto.” Domain of variables \( S \) and \( T \) = finite subsets of some universal set \( U \); domain of variable \( f \) = functions between subsets of \( U \); domain of \( x \) and \( y \) = \( U \). You may use the cardinality function \( |·| \) in any algebraic expression. Predicate: \( R(f, A, B) \), meaning “\( f \) is a function from \( A \) to \( B \)”. 

9. “For any function \( f : \mathbb{N} \rightarrow \mathbb{N} \), if there is some \( x \) so that \( f(x) > f(0) \), then there is some \( y < x \) so that \( f(y) < f(y+1) \).” Domain of variable \( f \) = \( \mathbb{N}^\mathbb{N} \); domain of variables \( x \) and \( y \) = \( \mathbb{N} \). 

10. “A function of the real numbers is continuous if and only if, for every \( x \) and \( \epsilon > 0 \), there is a \( \delta > 0 \) so that, if \( y \) is within \( \delta \) of \( x \), then \( f(y) \) is within \( \epsilon \) of \( f(x) \).” Domain of \( f \) = \( \mathbb{R}^\mathbb{R} \), domain of all other variables = \( \mathbb{R} \). Predicate: \( C(f) \), meaning “\( f \) is a continuous function.” 

11. “For any prime number \( p \), if \( p \) divides \( a \cdot b \), then \( p \) divides \( a \) or \( p \) divides \( b \).” Domain = \( \mathbb{Z} \). Predicates: \( P(n) \), meaning “\( n \) is prime” and \( D(a,b) \), meaning “\( a \) divides \( b \)”. 

Translations into Predicate Calculus: Solutions

1. \( \forall n(P(n) \leftrightarrow \forall k(D(k, n) \rightarrow ((k = 1) \lor (k = n)))) \)

2. \( \forall f \forall g(C(f) \land C(g) \rightarrow C(f + g)) \)

3. \( \forall x \forall y(Q(x) \land (x \neq 0) \land (x \neq 1) \land \neg Q(\sqrt{y}) \rightarrow \neg Q(x \sqrt{y})) \)

4. \( \forall a \forall b(\neg \exists c \exists d((a^b + 1 = c^d) \land (b \geq 2) \land (d \geq 2) \land \neg((a^b = 0) \land (c^d = 1)) \land \neg((a^b = 8) \land (c^d = 9))) \)

5. \( \forall d((d \geq 3) \rightarrow \neg \exists a \exists b \exists c(a^d + b^d = c^d)) \)

6. \( \forall a(D(4, a^2) \lor D(4, a^2 - 1)) \)

7. \( \forall a \forall b((Q(ab) \land (ab \neq 0)) \rightarrow (Q(a) \leftrightarrow Q(b))) \)

8. \( \forall S\forall T \forall f(R(f, S, T) \land (|S| = |T|)) \land \forall x \forall y((f(x) = f(y)) \rightarrow (x = y)) \rightarrow \forall x((x \in S) \rightarrow \exists y(f(y) = x)) \)

9. \( \forall f(\exists x(f(x) > f(0)) \rightarrow \exists y((y < x) \land (f(y) < f(y + 1)))) \)

10. \( \forall f(C(f) \leftrightarrow \forall x \forall \epsilon((\epsilon > 0) \rightarrow \exists \delta((\delta > 0) \land \forall y((|x - y| \leq \delta) \rightarrow (|f(x) - f(y)| \leq \epsilon)))))) \)

11. \( \forall p(P(p) \rightarrow \forall a \forall b(D(p, ab) \rightarrow (D(p, a) \lor D(p, b)))) \)