1. Show that

\[ [(p \rightarrow q) \rightarrow r] \rightarrow s \equiv [\neg (p \lor r) \lor (\neg r \land q) \lor s] \]

by (a) providing a truth table and (b) using a chain of logical equivalences.

(a)

|     | T | T | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | T | T | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | T | T | T | F | F | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | T | F | F | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | T | F | F | T | F | F | F | T | F | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | T | T | F | F | T | F | F | T | F | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | F | F | T | F | F | T | F | F | T | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | F | F | T | F | F | T | F | F | T | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | T | F | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | T | T | T | F | F | F | T | F | F | T | F | T | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | T | F | F | T | F | F | T | F | F | T | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | T | F | F | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | F | T | T | T | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | F | T | F | F | T | F | F | T | F | F | T | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | F | F | T | F | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |
| (p \rightarrow q) \rightarrow r | F | F | F | F | F | F | F | F | T | F | F | T | F | T | T | F | T | T | T | T | T | T | T | T | T |

(b)

\[(p \rightarrow q) \rightarrow r \rightarrow s \equiv \neg ((p \rightarrow q) \rightarrow r) \lor s \]

by the fact that \((a \rightarrow b) \equiv (\neg a \lor b)\)

\[\equiv \neg (\neg p \lor q) \lor r \lor s \]

by the fact that \((a \rightarrow b) \equiv (\neg a \lor b)\)

\[\equiv \neg \neg p \lor (\neg q) \lor r \lor s \]

by the fact that \((a \rightarrow b) \equiv (\neg a \lor b)\)

\[\equiv \neg (\neg p \lor (\neg q) \lor r) \lor s \]

by De Morgan’s Law

\[\equiv \neg (\neg (p \lor q) \lor r) \lor s \]

by the Law of Double Negation

\[\equiv (\neg (p \land \neg q) \land \neg r) \lor s \]

by De Morgan’s Law

\[\equiv (\neg (p \lor (\neg q) \land \neg r) \lor s \]

by De Morgan’s Law

\[\equiv (\neg (p \land q) \land \neg r) \lor s \]

by the Law of Double Negation

\[\equiv (\neg (p \land \neg r) \lor (q \land \neg r)) \lor s \]

by the Distributive Law

\[\equiv (\neg (p \lor r) \lor (q \land \neg r)) \lor s \]

by De Morgan’s Law

\[\equiv (\neg (p \lor r) \lor (\neg r \land q)) \lor s \]

by the Commutativity of Conjunction

\[\equiv (\neg (p \lor r) \lor (\neg r \land q) \lor s \]

by the Associativity of Disjunction

2. Translate the following statements into the Predicate Calculus, where the domain of all variables is the natural numbers (i.e., \(\mathbb{N} = \{0, 1, 2, \ldots\}\)), and the only predicate (other than algebraic expressions) you may use is \(D(a, b), \) meaning “\(a\) divides \(b\).”
(a) It is not true that there is a number strictly between any two given numbers.
\[ \neg \forall x \forall y ((x < y) \rightarrow \exists z ((x < z) \land (z < y))) \]

(b) The number \( k \) divides the number \( n \) if and only if there exists a number \( m \) so that \( n = km \).
\[ \forall k \forall n (D(k, n) \iff \exists m(n = km)) \]

(c) If \( a \) divides \( b \) and \( c \), then \( a \) divides \( b + c \).
\[ \forall a \forall b \forall c ((D(a, b) \land D(a, c)) \rightarrow D(a, b + c)) \]

More formally (so that we are not composing a predicate and a function):
\[ \forall a \forall b \forall c \forall d ((D(a, b) \land D(a, c) \land (d = b + c)) \rightarrow D(a, d)) \]

(d) For every \( n \), there is a prime larger than \( n \). (Recall that a prime number is a natural number which is divisible only by 1 and itself.)
\[ \forall n \exists p (p > n) \land \forall d (D(d, p) \rightarrow ((d = 1) \lor (d = p))) \]

3. Prove that \( 2^{2/3} \) is irrational. Do not assume anything except that every integer is either even or odd, and the ordinary rules of algebra!

We proceed by contradiction. Suppose that \( 2^{2/3} \) is not irrational, i.e., it can be written as \( 2^{2/3} = p/q \), where \( p \) and \( q \) share no factor other than 1. Then cubing both sides of this equation yields \( 4 = p^3/q^3 \), i.e., \( 4q^3 = p^3 \). Since the left-hand side is even (being twice the integer \( 2q^3 \)), the right hand side (\( p^3 \)) is as well. Now, suppose that \( p \) were odd. Then \( p = 2k+1 \) for some integer \( k \), so \( p^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2l + 1 \) with \( l = 4k^3 + 6k^2 + 3k \), and we may conclude that \( p^3 \) is odd. This is a contradiction, so \( p \) must be even. Then \( p \) can be written as \( p = 2p' \) for some integer \( p' \). Plugging this in to the above equation gives \( 4q^3 = (2p')^3 = 8p'^3 \), so that \( q^3 = 2p'^3 \). Since the right-hand side is now twice an integer (\( p'^3 \)), the left-hand side is even. But then, by the argument above, \( q \) must be even. This contradicts our hypotheses, since then \( p \) and \( q \) share a factor other than 1 (namely, 2). Therefore, we may conclude that \( 2^{2/3} \) is irrational. \( \square \)

4. Determine if the following statements are true or false. Either way, provide an (informal) proof of your answer. The domain of all variables is the nonnegative real numbers, and the predicate \( Q(x) \) means “\( x \) is rational”.

(a) \( \forall x \forall y ((x < y) \rightarrow \exists z (Q(z) \land (x < z) \land (z < y))) \)
True. Proof: Let \( m = \lfloor 2/(y - x) \rfloor > 0 \). Then

\[
x = \frac{1}{m} \cdot mx \\
\leq \frac{1}{m} \lfloor mx \rfloor \\
< \frac{1}{m} \left( \lfloor mx \rfloor + \frac{1}{2} \right) \\
< \frac{1}{m} \lfloor mx + 1 \rfloor \\
\leq \frac{1}{m} (mx + 2) \\
= x + \frac{2}{m} \\
= x + \frac{2}{\lfloor 2/(y - x) \rfloor} \\
\leq x + \frac{2}{2(y - x)} \\
\leq x + (y - x) = y.
\]

Therefore, if we let \( \alpha = \frac{1}{m} \left( \lfloor mx \rfloor + \frac{1}{2} \right) \), then \( x < \alpha < y \). Furthermore,

\[
\alpha = \frac{2 \lfloor mx \rfloor + 1}{2m},
\]

so \( \alpha \) is rational, since \( m \) is a (nonzero) integer and \( \lfloor \beta \rfloor \) is always an integer. Therefore, we may take \( z = \alpha \) to satisfy the claim. \( \square \)

(b) \( \forall x \exists y (y < x^2) \)

False. Proof: If we take \( x = 0 \), then \( x^2 = 0 \). However, there is no nonnegative \( y \) so that \( y < x^2 \), so the statement is false. \( \square \)

(c) \( \forall x ((x \neq 0) \rightarrow \exists y \forall z (xyz = z)) \)

True. Proof: If \( x \neq 0 \), we may define \( y = 1/x \). Then, for any \( z \), \( xyz = (xy)z = 1 \cdot z = z \). \( \square \)

(d) \( \forall x \forall y \left( (\neg Q(x) \land \neg Q(y)) \rightarrow \neg Q(x + y) \right) \)

False. Proof: Consider \( x = \sqrt{2} \) and \( y = 2 - \sqrt{2} \). Both of these are irrational nonnegative real numbers: \( x \) is irrational because we previously proved that it is, and \( y \) is irrational because, if it were not, i.e., \( y = p/q \) for some integers \( p \) and \( q \), then \( \sqrt{2} = 2 - y = 2 - p/q = (2q - p)/q \), contradicting the fact that \( \sqrt{2} \) is irrational. However, \( x + y = 2 \), which is clearly rational. \( \square \)

5. Prove that the following statements are equivalent for an integer \( n \):

(a) \( n \) is even.

(b) \( n^2 \) is even.

(c) There exists two odd integers \( a \) and \( b \) so that \( a + b = n \).

(d) If \( c \) and \( d \) are integers so that \( cd = n \), then \( c \) or \( d \) is even.

(e) For every integer \( k \), \( nk \) is even.
(a) → (b) If \( n = 2k \), then \( n^2 = 4k^2 = 2l \), where \( l = 2k^2 \). So, if \( n \) is even, then \( n^2 \) is even.

(b) → (a) Suppose \( n \) is odd, so \( n = 2k + 1 \) for some integer \( k \). Then \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2l + 1 \), where \( l = 2k^2 + 2k \), so \( n^2 \) is odd. Therefore, if \( n^2 \) is even, then \( n \) is even.

(a) → (c) Suppose \( n = 2k \) for some integer \( k \). Let \( a = n + 1 = 2k + 1 \) and \( b = -1 \). Clearly, \( a \) and \( b \) are odd and \( a + b = n \).

(c) → (a) Suppose \( a = 2k + 1 \) and \( b = 2l + 1 \) for some integers \( k \) and \( l \), and \( a + b = n \). Then

\[
   n = (2k + 1) + (2l + 1) = 2k + 2l + 2 = 2(k + l + 1) = 2m,
\]

where \( m = k + l + 1 \), so \( n \) is even.

(a) → (d) Suppose there exist two odd integers \( c \) and \( d \) so that \( cd = n \). Then \( c = 2k + 1 \) and \( d = 2l + 1 \) for some integers \( k \) and \( l \), so

\[
   n = cd = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2m + 1,
\]

where \( m = 2kl + k + l \). But then \( n \) is odd.

(d) → (a) Let \( c = 1 \) and \( d = n \). Then \( cd = n \), so either \( c \) is even or \( d \) is even. But \( c = 1 \) is odd, so we may conclude that \( d = n \) is even.

(a) → (e) If \( n = 2m \) for some integer \( m \), then for any integer \( k \), \( nk = 2mk = 2l \) with \( l = mk \). Therefore, \( nk \) is even.

(e) → (b) If we take \( k = n \), then we may conclude that \( nk = n^2 \) is even.

It is now clear that the transitive closure (i.e., all possible applications of hypothetical syllogism) yields full equivalence between all five properties. Indeed, this is evident from the following diagram of the above implications:

6. The harmonic mean of two nonzero real numbers \( x \) and \( y \) is the quantity \( 2/(1/x + 1/y) \), and their geometric mean is \( \sqrt{xy} \). Formulate a conjecture about the relative sizes of the harmonic and geometric means of two positive reals and then prove it.

**Proposition 1.** The harmonic mean is always less than or equal to the geometric mean of two positive real numbers.
Proof. Note that $(x - y)^2 \geq 0$, since the square of any real number is nonnegative. Therefore,

$$x^2 - 2xy + y^2 \geq 0.$$ 

Adding $4xy$ to both sides yields

$$x^2 + 2xy + y^2 \geq 4xy.$$ 

Now, the left hand side is just $(x + y)^2$. Multiplying both sides of this equation by the positive quantity $xy/(x + y)^2$ gives

$$xy \geq rac{4x^2y^2}{(x + y)^2}.$$ 

Now, since $x$ and $y$ are positive, both sides of this equation are nonnegative, so we may take square roots:

$$\sqrt{xy} \geq \sqrt{\frac{4x^2y^2}{(x + y)^2}} = \frac{2xy}{x + y} = \frac{2}{1/x + 1/y}.$$ 

Therefore, the geometric mean is always at least the harmonic mean. □