Problem 26 from Section 7.3 in Stewart:
Evaluate

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx.$$

Solution:
The denominator is a mess; we need to complete the square to get any closer to an integral that we might know how to evaluate. To find the appropriate constants, we collect terms according to the power of x:

$$\begin{aligned}3+4x-4x^2 &= A^2-(Bx+C)^2\\ &= (A^2-C^2)-2BCx-B^2x^2.\end{aligned}$$

This means, in particular, that $B=2$, since the coefficient of x^2 must agree on both sides of the equation. Furthermore,

$$2BC = 4C = -4,$$

by equating coefficients of the linear terms and plugging in $B=2$. Solving for C, this gives $C=-1$. By comparing the constant terms, we have

$$A^2-C^2 = A^2-1 = 3,$$

since we know that $C=-1$. Then $A=2$, and we can rewrite the integral as

$$\int \frac{x^2}{(4-(2x-1)^2)^{3/2}}\,dx.$$

Now, we make a substitution: $2x-1 = A\sin\theta = 2\sin\theta$. Solving for x gives

$$x = \sin\theta + \frac{1}{2}$$

Differentiating the substitution equation yields $2\,dx = 2\cos\theta\,d\theta$, i.e., $dx = \cos\theta\,d\theta$. Now, we may plug it in:

$$\begin{aligned}\int \frac{x^2}{(4-(2x-1)^2)^{3/2}}\,dx &= \int \frac{(\sin\theta+\frac{1}{2})^2}{(4-4\sin^2\theta)^{3/2}}\cos\theta\,d\theta\\ &= \int \frac{(\sin^2\theta+\sin\theta+\frac{1}{4})\cos\theta}{(4-4\sin^2\theta)^{3/2}}\,d\theta\\ &= \int \frac{(\sin^2\theta+\sin\theta+\frac{1}{4})\cos\theta}{4^{3/2}(1-\sin^2\theta)^{3/2}}\,d\theta\end{aligned}$$

$$= \int \frac{(\sin^2\theta + \sin\theta + \frac{1}{4})\cos\theta}{8(\cos^2\theta)^{3/2}}\,d\theta$$
$$= \int \frac{(\sin^2\theta + \sin\theta + \frac{1}{4})\cos\theta}{8\cos^3\theta}\,d\theta.$$

Now we may cancel a single factor of $\cos\theta$ from the numerator and denominator, and split up the integral:

$$\int \frac{\sin^2\theta + \sin\theta + \frac{1}{4}}{8\cos^2\theta}\,d\theta = \int \frac{\sin^2\theta}{8\cos^2\theta}\,d\theta + \int \frac{\sin\theta}{8\cos^2\theta}\,d\theta + \int \frac{1}{32\cos^2\theta}\,d\theta$$
$$= \frac{1}{8}\int \tan^2\theta\,d\theta + \frac{1}{8}\int \sec\theta\tan\theta\,d\theta + \frac{1}{32}\int \sec^2\theta\,d\theta.$$

The last two integrals come straight from our integral tables. The first one requires a touch more work, namely, using the trigonometric identity $\tan^2\theta = \sec^2\theta - 1$. This gives

$$\frac{1}{8}\int (\sec^2\theta - 1)\,d\theta + \frac{1}{8}\sec\theta + \frac{1}{32}\int \sec^2\theta\,d\theta$$
$$= -\frac{\theta}{8} + \frac{1}{8}\sec\theta + \left(\frac{1}{8} + \frac{1}{32}\right)\int \sec^2\theta\,d\theta$$
$$= -\frac{\theta}{8} + \frac{1}{8}\sec\theta + \frac{5}{32}\tan\theta + C.$$

Now we need to convert this back to an expression in terms of x. First of all, since $2x - 1 = 2\sin\theta$, we can solve for θ to get

$$\theta = \sin^{-1}(x - \frac{1}{2}).$$

In order to write $\sec\theta$ and $\tan\theta$ in terms of x, we need a right triangle. To have $\sin\theta = x - 1/2$, we make the length of the side opposite the angle θ equal to $x - 1/2$; the hypotenuse we just set to 1. (Recall that sin = opposite/hypotenuse.) By the Pythagorean Theorem, this makes the side adjacent to the angle θ have length

$$\sqrt{1^2 - \left(x - \frac{1}{2}\right)^2} = \sqrt{1 - \left(x^2 - x + \frac{1}{4}\right)} = \sqrt{\frac{3}{4} + x - x^2}.$$

Then

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\sqrt{\frac{3}{4} + x - x^2}},$$

and

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x - \frac{1}{2}}{\sqrt{\frac{3}{4} + x - x^2}} = \frac{2x - 1}{\sqrt{3 + 4x - 4x^2}}.$$

Putting the pieces together, we have the answer

$$\begin{aligned}
\int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}}\, dx &= -\frac{\sin^{-1}\left(x - \frac{1}{2}\right)}{8} + \frac{1}{8\sqrt{\frac{3}{4} + x - x^2}} \\
&\qquad + \frac{10x - 5}{32\sqrt{3 + 4x - 4x^2}} + C \\
&= -\frac{\sin^{-1}\left(x - \frac{1}{2}\right)}{8} + \frac{1}{4\sqrt{3 + 4x - 4x^2}} \\
&\qquad + \frac{10x - 5}{32\sqrt{3 + 4x - 4x^2}} + C \\
&= -\frac{\sin^{-1}\left(x - \frac{1}{2}\right)}{8} + \frac{8}{32\sqrt{3 + 4x - 4x^2}} \\
&\qquad + \frac{10x - 5}{32\sqrt{3 + 4x - 4x^2}} + C \\
&= -\frac{\sin^{-1}\left(x - \frac{1}{2}\right)}{8} + \frac{10x + 3}{32\sqrt{3 + 4x - 4x^2}} + C.
\end{aligned}$$