1. Approximate the definite integral \( \int_{2}^{3} \frac{1}{x} \, dx \) by taking a Riemann sum with \( n = 3 \) equal-length subintervals using the left-hand endpoints.

**Solution:** We let \( a = 2, \ b = 3, \ n = 3, \ f(x) = \frac{1}{x} \), \( \Delta x = (b - a)/n = 1/3 \), so that \( x_0 = 2, \ x_1 = 2 + \frac{1}{3} = 7/3, \ x_2 = 2 + \frac{2}{3} = 8/3, \ x_3 = 3 \). Then the appropriate Riemann sum is

\[
\sum_{i=0}^{2} f(x_i) \Delta x = \frac{1}{3} \left( \frac{1}{2} + \frac{1}{7/3} + \frac{1}{8/3} \right) \\
= \frac{1}{3} \left( \frac{1}{2} + \frac{3}{7} + \frac{3}{8} \right) \\
= \frac{1}{3} \cdot \frac{73}{56} = \frac{73}{168}.
\]

2. Compute the definite integral \( \int_{0}^{\pi/4} \sec x \tan x \, dx \).

**Solution:** Let \( f(x) = \sec x \tan x \). By the Fundamental Theorem of Calculus,

\[
\int_{0}^{\pi/4} f(x) \, dx = F(\pi/4) - F(0),
\]

where \( F \) is any antiderivative of \( f \). Since \( \frac{d}{dx} \sec x = \sec x \tan x \), we may take \( F(x) = \sec x \). Then

\[
\int_{0}^{\pi/4} f(x) \, dx = \sec(\pi/4) - \sec(0) = \frac{1}{\cos(\pi/4)} - \frac{1}{\cos(0)} = \sqrt{2} - 1.
\]

3. Compute \( \lim_{x \to \infty} xe^{1/x} - x \).

**Solution:** This is an indeterminate limit of the form \( \infty - \infty \), so we must rewrite it as a ratio in order to apply L'Hôpital's Rule. To that end,

\[
x e^{1/x} - x = x(e^{1/x} - 1) = \frac{e^{1/x} - 1}{1/x}.
\]
Now, as \( x \to \infty \), the numerator and denominator of this fraction approach zero, so it is a limit of indeterminate form 0/0 and we may apply L'Hôpital’s Rule. Differentiating the numerator and denominator separately yields

\[
\lim_{x \to \infty} \frac{e^{1/x} - 1}{1/x} = \lim_{x \to \infty} \frac{-x^{-2}e^{1/x}}{-x^{-2}} = \lim_{x \to \infty} \frac{e^{1/x}}{1} = e^0 = 1.
\]

4. Compute

\[
\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{3x^2 + x}.
\]

**Solution:** This is an indeterminate limit of the form 0/0, since \( \lim_{x \to 0} (e^{2x} - 1 - 2x) = 1 - 1 - 0 = 0 \) and \( \lim_{x \to \infty} (3x^2 + x) = 0 + 0 = 0 \), so we may apply L'Hôpital’s Rule directly:

\[
\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{3x^2 + x} = \lim_{x \to 0} \frac{2e^{2x} - 2}{6x + 1} = \frac{2 \cdot e^0 - 2}{6 \cdot 0 + 1} = \frac{0}{1} = 0.
\]

5. What is \( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \frac{2j}{n} - 1 \right)^2 \)? (Your answer should be a number.)

**Solution:** This is clearly a Riemann sum converging to a certain definite integral. Call the quantity requested \( S \). Note that the quantity being squared varies from (when \( j = 0 \), which is not actually included in the sum because it uses the right-hand endpoints) \(-1\) to (when \( j = n \)) \(2n/n - 1 = 2 - 1 = 1\). If the distance between these two quantities — i.e., \( b - a = 1 - (-1) = 2 \) — is cut into \( n \) pieces of width \( \Delta x \), then \( \Delta x = (b - a)/n = 2/n \). Therefore, we may write

\[
\int_{-1}^{1} x^2 \, dx = \lim_{n \to \infty} \frac{2}{n} \sum_{j=1}^{n} \left( \frac{2j}{n} - 1 \right)^2 = 2S.
\]
Then, applying the Fundamental Theorem of Calculus,

\[ S = \frac{1}{2} \int_{-1}^{1} x^2 \, dx \]

\[ = \frac{1}{2} \left[ \frac{x^3}{3} \right]_{-1}^{1} \]

\[ = \frac{1}{2} \left( \frac{1}{3} - \frac{(-1)^3}{3} \right) \]

\[ = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}. \]

6. Acme Cheap Housing Co. is building a box-shaped house with a budget of $30,000. The material for the walls costs $10/m², the material for the (flat) roof costs $20/m², and the material for the floor costs $5/m². What is the largest volume of a house they can build? (You may ignore windows, doors, etc.)

**Solution:** Suppose the house to be built has height \( h \), width \( w \), and depth \( d \). Then its volume is given by \( V = hwd \). Its cost of construction, on the other hand, is given by \( 30000 = 10(2w + 2d)h + (20 + 5)wd = 10Ph + 25A \), where \( P \) is the perimeter of the floor and \( A \) is the area of the floor. As we showed in class, the rectangle of smallest perimeter for a given area is a square; therefore, whatever Acme decides the floor’s area must be, they will choose \( w = d \) to bring the total cost to a minimum. Using this equality allows us to rewrite the volume equation as \( V = hw^2 \) and the cost of construction as \( 30000 = 40wh + 25w^2 \). Solving the latter for \( h \) yields

\[ h = \frac{30000 - 25w^2}{40w} = \frac{6000 - 5w^2}{8w}. \]

Plugging this into the volume gives

\[ V(w) = \frac{6000 - 5w^2}{8w} \cdot w^2 = \frac{6000w}{8} - \frac{5w^3}{8} = 750w - \frac{5w^3}{8}. \]

In order to maximize \( V \), we must find out which \( w \geq 0 \) gives the largest volume. Note that \( V(w) \) must be nonnegative, so \( 750w - 5w^3/8 \geq 0 \), i.e., \( 1200 \geq w^2 \), since \( w \geq 0 \). Then \( w \leq \sqrt{1200} = 20\sqrt{3} \). Therefore, we wish to determine the maximum value of the continuous function \( V(w) \)
on the interval $[0, 20\sqrt{3}]$, which the Extreme Value Theorem guarantees exists. Plugging in $w = 0$ gives a volume of 0, and plugging in $w = 20\sqrt{3}$ gives $V(w) = 0$ as well. To find the critical points of $V(w)$, we compute the derivative:

$$\frac{dV}{dw} = 750 - \frac{15w^2}{8}.$$ 

This is equal to zero when $750 = 15w^2/8$, i.e., $w = \pm 20$. Only the positive value for $w$ makes sense, so there is a critical point at $w = 20$. In that case, $V(w) = V(20) = 750 \cdot 20 - 5 \cdot (20)^3/8 = 15000 - 5000 = 10000$. Since this value is clearly larger than the value of $V(w)$ at the endpoints of the interval under consideration, the largest possible volume is indeed 10000m$^3$.

7. A ball is thrown up in the air with a vertical velocity of 64 ft/s and a horizontal velocity of 24 ft/s from the floor of a deep hole in the ground. Assuming it just barely clears the edge of the hole (and assuming air friction, the thrower’s height, etc., are negligible), how far away from the thrower is the ball when it hits the ground? (See diagram below.)

\[ \text{Solution:} \] As always, gravitational acceleration on the surface of the Earth is given by $a(t) = -32t/s^2$. Antidifferentiating the acceleration gives the (vertical) velocity $v(t) = -32t + C_1$, and antidifferentiating the velocity gives the height of the ball $h(t) = -16t^2 + C_1t + C_2$, for some constants $C_1$ and $C_2$. 

![Diagram of a ball thrown up in the air with a vertical velocity of 64 ft/s and a horizontal velocity of 24 ft/s from the floor of a deep hole in the ground. The ball just barely clears the edge of the hole, and the thrower is asked how far away from the thrower the ball is when it hits the ground.]
The vertical velocity at the beginning is given by $v(0) = 64$, so $C_1 = 64$ and we may write $h(t) = -16t^2 + 64t + C_2$. The fact that the ball just barely clears the edge of the hole means that, when $v(t) = 0$ for some time $t$ which we will refer to as $t_{\text{top}}$, we know that the height of the ball is exactly 0 (since it is at ground level), i.e., $h(t_{\text{top}}) = 0$. To find $t_{\text{top}}$, we write $v(t_{\text{top}}) = -32t_{\text{top}} + 64 = 0$, so $t_{\text{top}} = 2$. Therefore, $0 = h(2) = -16 \cdot 2^2 + 64 \cdot 2 + C_2 = -64 + 128 + C_2 = 64 + C_2$. Then $C_2 = -64$. Since $-h(0) = -(-64) = 64$ is the depth of the hole, we may conclude that the thrower is 64ft below ground, and that the ball strikes the edge of the hole after two seconds of flight. Therefore, the point where the ball lands is a vertical distance of 64ft away, and a horizontal distance of $24\text{ft/s} \cdot 2\text{s} = 48\text{ft}$ away. Then the distance from the thrower to the edge of the hole is given by the Pythagorean Theorem:

$$\sqrt{64^2 + 48^2} = 16\sqrt{4^2 + 3^2} = 16\sqrt{25} = 16 \cdot 5 = 80\text{ft}.$$