Characteristic Power Series of Graph Limits

Joshua N. Cooper*

September 9, 2020

Abstract

In this note, we show how to obtain a "characteristic power series" of graphons – infinite limits of dense graphs – as the limit of normalized reciprocal characteristic polynomials. This leads to a new characterization of graph quasi-randomness and another perspective on spectral theory for graphons, a complete description of the function in terms of the spectrum of the graphon as a self-adjoint kernel operator. Interestingly, while we apply a standard regularization to classical determinants, it is unclear how necessary this is.

1 Introduction

A research direction began in the 1980's with graph quasi-randomness, extended through the 1990's and early 2000's with generalizations to non-uniform graph distributions and other combinatorial objects, became graph limit theory in the mid-2000's, and culminated in Lovász's now-canonical text [13]. The central idea is that, if a sequence of graphs G_n with number of vertices tending to infinity has the property that the density of any particular subgraph tends to a limit, then G_n itself tends to a limit object $\mathcal{G}$, called a "graphon". There are several mutually (though non-obviously) equivalent ways to view graphons, and a central one is as a self-adjoint kernel operator from $L^1([0,1])$ to $L^\infty([0,1])$, an object type for which a well-established spectral theory exists. In particular, a graphon, when thought of as this kernel, is a symmetric function $[0,1] \times [0,1] \to [0,1]$, up to composition of both coordinates with a measure-preserving bijection of $[0,1]$ and up to modification on a set of measure 0. Indeed, we use the same notation throughout for $\mathcal{G}$ as well as (any representative of the equivalence class of) its kernel. There is also a key notion of subgraph density for graphons, with the property that densities in convergent sequences of graphs in a sequence converge to their densities in the limit graphon, often referred to as "left-convergence". Furthermore, Szegedy ([18]) introduced a spectral theory of graphons by studying the eigenpairs of their kernels and showed that it is a natural analogue of the spectral theory of finite graph adjacency matrices. Here, we extend this

*Department of Mathematics, University of South Carolina, Columbia, SC USA

perspective by showing that graphons are associated with a power series which is a certain normalized limit of the characteristic polynomial of graphs. Furthermore, we give an equivalent definition of this "characteristic power series" $\psi_{\mathcal{G}}(z)$ of a graphon $\mathcal{G}$ via a regularized determinant of its corresponding kernel function.

We also show that the characteristic power series can be used to characterize "quasi-randomness". Suppose that $\{G_n\}_{n\geq 1}$ is a sequence of graphs with $|V(G_n)| = n$. (In truth, all that is needed is that $|G_n| \to \infty$, but this is not more general.) We write $G = G_n$ for simplicity and $(\#H \subseteq G)$ for the number of labelled, not-necessarily induced copies of H as subgraphs in G (i.e., injective homomorphisms from H to G). Then, by a classic 1989 paper of Chung, Graham, and Wilson ([6]), there is a large set of random-like properties (properties which hold asymptotically almost surely for graphs in the Erdős-Rényi model $G(n,p)$) which are mutually equivalent, and are therefore collectively referred to as (the sequence of graphs) G being "quasi-random". Namely, let

- $P_1(s)$ denote the property that the number of labelled occurrences of each graph on s vertices as an induced subgraph of G is $(1+o(1))n^s p^{|E(H)|}(1-p)^{\binom{s}{2}-|E(H)|}$
- $P_1'(s)$ denote the property that $(\#H \subset G) = (1+o(1))n^s p^{|E(H)|}$ for each graph H on s vertices
- $P_2(t)$ denote the property that $|E(G)| \geq (1+o(1))pn$ and $(\#C_t \subseteq G) \leq (1+o(1))(np)^t$, where C_t is the t-cycle
- P_3 denote the property that $|E(G)| \geq (1+o(1))pn$, $\lambda_1 = pn(1+o(1))$, and $\lambda_2 = o(n)$, where $|\lambda_1| > \cdots > |\lambda_n|$ are the complete set of adjacency eigenvalues of G
- P_4 denote the property that, for all $S \subseteq V(G)$, $E(G[S]) = p^2|S|^2 + o(n^2)$, where $G[S]$ denotes the subgraph of G induced by S
- P_5 denote the property that, for all $S \subseteq V(G)$ with $|S| = \lfloor n/2 \rfloor$, $E(G[S]) = p^2|S|^2/4 + o(n^2)$, where $G[S]$ denotes the subgraph of G induced by S
- P_6 denote the property that

  $$\sum_{v,w\in V(G)} ||N_G(v) \triangle N_G(w)| - 2np(1-p)| = o(n^3),$$

  where $\triangle$ denotes symmetric difference
- P_7 denote the property that

  $$\sum_{v,w\in V(G)} \big||N_G(v) \cap N_G(w)| - np^2\big| = o(n^3)$$

Theorem 1 (Chung-Graham-Wilson [6]). *For $s \geq 4$ and $t \geq 4$ even, and any fixed $p \in [0,1]$,*

$$P_2(4) \Leftrightarrow P_2(t) \Leftrightarrow P_1(s) \Leftrightarrow P_1'(s) \Leftrightarrow P_3 \Leftrightarrow P_4 \Leftrightarrow P_5 \Leftrightarrow P_6 \Leftrightarrow P_7$$

If a graph sequence G has these properties, it is called p-quasi-random, and these properties and any others also equivalent to them are known as (p-)quasi-random properties. Many other quasi-random properties have been added since to the list above, such as other families $\mathcal{F}$ of graphs whose occurrence as subgraphs at the "random-like rate" implies these properties (note that $\{K_2, C_4\}$ is the "forcing" family given by $P_2(4)$), and also that G_n converges to a constant graphon.

Here, we propose to add another property. First, given a left-convergent sequence of graphs G, let $\mathcal{G}$ be their limit, and let $\phi_n \in \mathbb{C}[x]$ denote the (adjacency) characteristic polynomials of G_n. Recall that the characteristic polynomial of a graph G with adjacency matrix A is defined to be $\phi(x) = \det(Ix - A)$; it is easy to see that ϕ is monic and has only real roots. Define

$$\psi_{\mathcal{G}}(z) = \begin{cases} \lim_{n\to\infty} \left(x^{-n}\phi_n(x)\right)\Big|_{x=n/z} & \text{if } z \neq 0 \\ 1 & \text{otherwise} \end{cases},$$

if the (pointwise) limit exists. We call $\psi_{\mathcal{G}}(z)$ the "characteristic power series" of the graphon $\mathcal{G}$, and it is essentially a normalized limit of the reciprocal polynomials of the characteristic functions of G. In Theorem 3 below, we show that $\psi_{\mathcal{G}}$ is indeed well-defined, i.e., independent of the sequence of graphs left-converging to $\mathcal{G}$.

2 Characteristic Power Series of Graphons

The following classical result will be useful in describing the coefficients of $\psi_{\mathcal{G}}(z)$.

Theorem 2 (Harary-Sachs [12]). *Suppose G is a graph on n vertices, and $k \geq 0$ is an integer. The coefficient of x^{n-k} in $\phi_G(x)$ is*

$$\sum_{H \in \mathcal{H}_k} (-1)^{c(H)} 2^{z(H)} [\#H \subseteq G]$$

where $c(H)$ is the number of components of H, $z(H)$ is the number of cycles of H, and $\mathcal{H}_k$ is the family of all unlabelled graphs on k vertices each of whose components is an edge or a cycle, and $[\#H \subseteq G]$ denotes the number of subsets of edges of G which are isomorphic to H. (When $k = 0$, $\mathcal{H}_k$ is the singleton consisting only of the empty graph ϵ; take $c(\epsilon) = z(\epsilon) = 0$.)

Clearly, if $H \in \mathcal{H}_k$, then $|E(H)| = k - c(H) + z(H)$. Also,

$$(\#H \subseteq G) = [\#H \subseteq G] \cdot 2^{z(H)} \prod_i a_i^{m_i} m_i!$$

where H has m_i components of size a_i for each i. For simplicity, for a partition λ where part b_i occurs m_i times for each i (the b_i all distinct), the quantity $\eta(\lambda)$ is defined by

$$\eta(\lambda) := \prod_i b_i^{m_i} m_i!$$

and we write $\eta(H) = \eta(\lambda(H))$ where $\lambda(H)$ is the integer partition of $|E(H)|$ given by the component cardinalities of H. If λ is a partition of n, then the number of partitions of an n-set with structure λ (a partition with m_i parts of distinct sizes b_i) is given by

$$\frac{n!}{b_1!^{m_1} \cdots b_t!^{m_t} m_1! \cdots m_t!} = \frac{n!}{\eta(\lambda) \prod_{i=1}^t (b_i - 1)!^{m_i}}$$

Denote by $\Lambda'_{n,k}$ the set of partitions of n into k parts, each of which is of size at least 2; for $\lambda \in \Lambda'_{n,k}$, denote its i-th largest part by λ_i, its i-th largest part size by b_i, and the multiplicity of b_i by m_i.

Letting $z = n/x$, we have by Theorem 2,

$$\begin{aligned} x^{-n}\phi_{G_n}(x) &= x^{-n} \sum_{k=0}^{n} x^{n-k} \sum_{H \in \mathcal{H}_k} (-1)^{c(H)} 2^{z(H)} [\# H \subseteq G] \\ &= \sum_{k=0}^{n} \sum_{\lambda \in \Lambda'_{n,k}} x^{-k} (-1)^k \frac{(\# \bigcup_i C_{\lambda_i} \subseteq G)}{\eta(\lambda)} \\ &= \sum_{k=0}^{n} \sum_{\lambda \in \Lambda'_{n,k}} z^k (-1)^k \frac{(\# \bigcup_i C_{\lambda_i} \subseteq G)}{n^k \prod_i b_i^{m_i} m_i!}. \end{aligned}$$

Note that (treating this as a polynomial to avoid defining 0^0), when $z = 0$, the above expression equals 1 because $\phi_G(x)$ is monic. Denote this polynomial by $\psi_n(z)$. Write $t(H, G)$ for the "homomorphism density" of the k-vertex graph H in the n-vertex G, i.e., the number of (not necessarily injective) homomorphisms from H to G over n^k. Writing λ_i for the i-th eigenvalue of $A(G)/n$, we may bound

$$\begin{aligned} |\psi_n(z)| &\leq \sum_{k=0}^{n} z^k \sum_{\lambda \in \Lambda'_{n,k}} \frac{t(\bigcup_i C_{\lambda_i}, G)}{\prod_i b_i^{m_i} m_i!} \\ &= \sum_{k=0}^{n} \sum_{\lambda \in \Lambda'_{n,k}} \prod_{i=1}^{k} \frac{z^{m_i} t(C_{b_i}, G)^{m_i}}{b_i^{m_i} m_i!} \\ &= \prod_{b \geq 2} \exp\left(\frac{z t(C_b, G)}{b}\right) = \exp\left(\sum_{b \geq 2} \frac{z t(C_b, G)}{b}\right) \\ &= \exp\left(\sum_{b \geq 2} \frac{z \sum_i \lambda_i^b}{b}\right) = \exp\left(-z \sum_i [\log(1 - \lambda_i) + \lambda_i]\right) \end{aligned}$$

$$= \left[\prod_i (1-\lambda_i)\right]^{-z} \cdot \exp\left(-z\sum_i \lambda_i\right) = \left[\prod_i (1-\lambda_i)\right]^{-z}$$

since $\sum_i \lambda_i = \operatorname{tr} A(G)/n = 0$. The above product converges if $\sum_i |\lambda_i|$ does, which is n^{-1} times the so-called "energy" $\mathcal{E}(G)$ of G, the sum of its adjacency singular values. Since $\mathcal{E}(G)$ can be as large as $Cn^{3/2}$, we should not hope for $\phi_n(z)$ always to converge. Indeed, $n^{3/2}$ tends to be the order of magnitude of the energy of dense graphs, i.e., graphs with $\Omega(n^2)$ edges, the only graphs converging to a nontrivial graphon; see, for example, [14]. However, this is not *always* the case: indeed, $\mathcal{E}(K_n) = 2n-2$.

Therefore, we must introduce the so-called "regularized characteristic determinant" $\det^{(p)}(\mathcal{A})$ of a linear operator $\mathcal{A}$:

Definition 1. *The* regularized characteristic determinant *of a linear operator $\mathcal{A}$ is defined as*

$$\overset{(p)}{\det}(I - z\mathcal{A}) = \prod_j \left[(1-\lambda_j(\mathcal{A})z)\exp\left(\sum_{k=1}^{p-1} \lambda_j^k z^k/k\right)\right]$$

where λ_j varies over the eigenvalues of $\mathcal{A}$.

We then use this definition – albeit only the $p = 2$ case, a.k.a. the Hilbert-Carleman determinant – for reasons which will be apparent below, to define a characteristic power series of graphons:

Definition 2. *The* characteristic power series *of a graphon $\mathcal{G}$ is defined by*

$$\psi_{\mathcal{G}}(z) = \overset{(2)}{\det}(I - z\mathcal{G})\exp\left(z^2 \cdot \frac{\|\mathcal{G}\|_2^2 - \|\mathcal{G}\|_1}{2}\right).$$

By [10] (Chapter IV, Section 2), the function $\det^{(p)}(I - z\mathcal{A})$ is well-defined and entire (of genus $p-1$) for operators $\mathcal{A}$ in $\mathfrak{S}_p$, the operators which are Schatten p-class, i.e., for which the Schatten p-norm $(\sum_i \sigma_i^p)^{1/p}$ is finite, where σ_i are the singular values of $\mathcal{A}$, defined to be the eigenvalues of $\sqrt{\mathcal{A}^*\mathcal{A}}$. Since graphons give rise to self-adjoint operators, we will have throughout that $\sigma_i = |\lambda_i|$. Note that Schatten 2-class bounded operators are the same as Hilbert-Schmidt operators, which all graphons' corresponding integral transforms are; and Schatten 1-class are the nuclear or trace-class operators, in which case $\det^{(1)}(\mathcal{A})$ is the classical Fredholm determinant and $\operatorname{tr}(\mathcal{A}) = \sum_j \lambda_j$ is the (signed) trace. It also follows from [10] (see Theorem IV.2.1) that $\det^{(p)}(I - z\mathcal{A})$ is continuous (uniform convergence on compact sets) with respect to convergence in p-norm of $\mathcal{A}$.

We now present our main theorem, demonstrating that $\psi_{\mathcal{G}}$ is indeed well-defined and is an entire function of Laguerre-Pólya class, i.e., a holomorphic function which is locally the limit of a series of polynomials whose roots are all real. Laguerre-Pólya functions have played a prominent role in the study of distributions of zeros of real polynomials and real entire functions (e.g., [2]), early

20th-century attempts to prove the Riemann hypothesis and a recent revival of such methods (see [11]), and classical complex analysis. The fact that $\psi_{\mathcal{G}}(z)$ is Laguerre-Pólya class implies that it has a Hadamard product expression (see, e.g., [3] Theorem 2.7.1):

$$\psi_{\mathcal{G}}(z) = z^m \exp(a + bz + cz^2) \prod_r \left(1 - \frac{z}{r}\right) \exp\left(\frac{z}{r}\right) \tag{1}$$

where m is a nonnegative integer; b and c are real with $c \leq 0$; and r ranges over the nonzero zeros of $\psi_{\mathcal{G}}(z)$. Note that the definition of $\psi_{\mathcal{G}}(z)$ is almost in this form already. In particular, $m = a = b = 0$, and $c = (\|\mathcal{G}\|_2^2 - \|\mathcal{G}\|_1)/2$, and the product ranges over the reciprocals r of the nonzero eigenvalues of $\mathcal{G}$.

Theorem 3. *Suppose the graphs G_n converge to the graphon $\mathcal{G}$. For the sequence of functions $\psi_n(z)$ corresponding to the sequence of graphs G_n:*

1. *$\psi_n(z)$ converges pointwise as $n \to \infty$.*
2. *$\psi_n(z)$ converges uniformly on compact sets as $n \to \infty$.*
3. *Each coefficient of $\psi_n(z)$ converges as $n \to \infty$.*

Furthermore, the limit is $\psi_{\mathcal{G}}(z)$, is entire of Laguerre-Pólya class, and its roots are the reciprocals of the nonzero eigenvalues of $\mathcal{G}$ (as a self-adjoint kernel operator) with multiplicity.

Proof. Let $A' = A(G)/n$, so that

$$\begin{aligned}\psi_n(z) &= \left(x^{-n} \det(xI - A(G))\right)\big|_{x=n/z} \\ &= \left(\det(I - A(G)x^{-1})\right)\big|_{x=n/z} \\ &= \det(I - A'z).\end{aligned}$$

As shown in [18] (Section 1.4), the (modulus-ordered) spectrum of A' converges to that of $\mathcal{G}$ in ℓ^4. Therefore, denoting the eigenvalues of A' by $\{\lambda_i\}_{i=1}^n$,

$$\prod_j \left(1 - \frac{\lambda_j(A)z}{n}\right) \exp\left(\sum_{k=1}^{3} \frac{\lambda_j(A)^k z^k}{k}\right) \to \prod_j (1 - z\lambda_j(\mathcal{G})) \exp\left(\sum_{k=1}^{3} \frac{\lambda_j(\mathcal{G})^k z^k}{k}\right) \tag{2}$$

uniformly on compact sets. Since $\mathcal{G}$ is $\mathfrak{S}_2$-class, the function $\det^{(2)}(\mathcal{G})$ is defined and entire, so the right-hand side of (2) can be written

$$\overset{(2)}{\det}(I - z\mathcal{G}) \prod_j \exp\left(\frac{\lambda_j(\mathcal{G})^2 z^2}{2} + \frac{\lambda_j(\mathcal{G})^3 z^3}{3}\right)$$

Similarly, the left-hand side of (2) can be written

$$\psi_n(z) \prod_j \exp\left(\frac{\lambda_j(A)^2 z^2}{2} + \frac{\lambda_j(A)^3 z^3}{3}\right)$$

because $\sum_j \lambda_j = \mathrm{tr}(A') = 0$. There is a natural definition of the homomorphism densities $t(F, \mathcal{G})$ in terms of a certain integral which is standard for graphons, and [13] (Theorem 7.22) shows that $\sum_j \lambda_j^k = t(C_k, \mathcal{G})$ for $k \geq 2$ and that $t(C_k, G_n) \to t(C_k, \mathcal{G})$ for $k \geq 3$. Thus, the cubic terms can be cancelled in (2) and the quadratic terms behave predictably, because $\sum_j \lambda_j^2 = \|\mathcal{G}\|_2^2$ (the Hilbert-Schmidt norm) and $t(C_2, G_n) = 2|E(G_n)|/n^2$:

$$\psi_n(z) \exp\left(\frac{z^2 |E(G_n)|}{n^2}\right) \to \overset{(2)}{\det}(I - z\mathcal{G}) \exp\left(\frac{z^2 \|\mathcal{G}\|_2^2}{2}\right)$$

Since the edge density $\sum_j \lambda_j(A')^2 = t(C_2, G_n) = 2\|E(G_n)\|/n^2$ converges to $\|\mathcal{G}\|_1$, this can be rewritten as

$$\lim_{n\to\infty} \psi_n(z) = \overset{(2)}{\det}(I - z\mathcal{G}) \exp\left(z^2 \cdot \frac{\|\mathcal{G}\|_2^2 - \|\mathcal{G}\|_1}{2}\right) = \psi_{\mathcal{G}}(z)$$

where the limit can be interpreted as uniform convergence on compact subsets of $\mathbb{C}$ or pointwise.

Then (1) and (2) follow, and, since (2) holds, Cauchy's integral formula implies that (3) holds as well. Since $\psi_n(z)$ has only real roots (being the characteristic polynomial of a real symmetric matrix), the limit is of Laguerre-Pólya class. That the roots of $\psi_{\mathcal{G}}(z)$ are the eigenvalues of the kernel operator corresponding to $\mathcal{G}$ is a consequence of Corollary 6.3 of [16], which states that $\det^{(p)}(I + \mathcal{A}) \neq 0$ iff $I + \mathcal{A}$ is invertible. □

Theorem 3 has immediate consequences from various properties of determinants, for example the following result. Here we introduce the notation $\mathcal{G} \oplus_p \mathcal{H}$ for the *p-disjoint union* of $\mathcal{G}$ and $\mathcal{H}$, the graphon whose kernel W is given by

$$W(x, y) = \begin{cases} \mathcal{G}\left(\frac{x}{p}, \frac{y}{p}\right) & \text{if } x \in [0, p) \\ \mathcal{H}\left(\frac{x-p}{1-p}, \frac{y-p}{1-p}\right) & \text{if } y \in [p, 1] \end{cases}.$$

Corollary 1. *Given two graphons $\mathcal{G}$ and $\mathcal{H}$, the graphon $\mathcal{G} \oplus_p \mathcal{H}$ which is their disjoint union has the property that*

$$\psi_{\mathcal{G}\oplus\mathcal{H}}(z) = \psi_{\mathcal{G}}(pz)\psi_{\mathcal{H}}((1-p)z).$$

Proof. The kernel of $\mathcal{G} \oplus_p \mathcal{H}$ is $\mathcal{G}' + \mathcal{H}'$, where $\mathcal{G}'(x, y) = \mathcal{G}(x/p, y/p)$ and $\mathcal{H}'(x, y) = \mathcal{H}((x - p)/(1 - p), (x - p)/(1 - p))$ (interpreting functions to be zero outside $[0, 1] \times [0, 1]$). By [10] (Section VI.2) and the fact that $\mathcal{G}'\mathcal{H}' = 0$ (multiplication interpreted as composition), the Hilbert-Carleman determinant satisfies

$$\overset{(2)}{\det}(I - z\mathcal{G} \oplus \mathcal{H}) = \overset{(2)}{\det}((I - z\mathcal{G}')(I - z\mathcal{H}')) = \overset{(2)}{\det}(I - z\mathcal{G}) \overset{(2)}{\det}(I - z\mathcal{H}) e^{-z \sum_i \lambda_i(\mathcal{G}'\mathcal{H}')}.$$

But $\mathcal{G}'\mathcal{H}' = 0$, so $\sum_i \lambda_i(\mathcal{G}'\mathcal{H}') = 0$. Then

$$\psi_{\mathcal{G}\oplus\mathcal{H}}(z) = \overset{(2)}{\det}(I - z\mathcal{G} \oplus \mathcal{H}) \exp\left(z^2 \cdot \frac{\|\mathcal{G} \oplus \mathcal{H}\|_2^2 - \|\mathcal{G} \oplus \mathcal{H}\|_1^1}{2}\right)$$

$$= \overset{(2)}{\det}(I - z\mathcal{G}')\overset{(2)}{\det}(I - z\mathcal{H}')$$

$$\cdot \exp\left[\frac{z^2}{2}\left(p^2\|\mathcal{G}'\|_2^2 + (1-p)^2\|\mathcal{H}'\|_2^2 - p^2\|\mathcal{G}'\|_1 - (1-p)^2\|\mathcal{H}'\|_1\right)\right]$$

$$= \psi_{\mathcal{G}'}(z)\psi_{\mathcal{H}'}(z) = \psi_{\mathcal{G}}(pz)\psi_{\mathcal{H}}((1-p)z).$$

□

We now apply our main result to give another characterization of quasi-random graphs.

Theorem 4. *The property of a sequence of graphs G_n that $\psi_n(z)$ converges pointwise to a function with only one root (of multiplicity one) at $z = 1/p$ is a p-quasi-random property for $p \in (0,1]$. For $p = 0$, it is equivalent to p-quasi-randomness that $\psi_n(z)$ converges to the constant function* 1.

Proof. Suppose $p > 0$. Let $\mathcal{G}$ be the p-constant graphon, i.e., the limit of a p-quasirandom graph sequence. By Theorem 3 and Hurwitz's Theorem, coefficient convergence implies root convergence, in the sense that every ϵ ball, for $\epsilon > 0$ sufficiently small, about a zero of $\psi_{\mathcal{G}}(z)$ of multiplicity m will contain exactly m roots of $\phi_n(z)$ for sufficiently large n, including for roots at infinity. If G is p-quasirandom, then by P_3 the eigenvalues of G_n are $pn + o(n)$ (once) and $o(n)$ (with multiplicity $n-1$), so the roots of $\psi_n(z)$ are $1/p + o(1)$ (once) and some $n-1$ roots the smallest modulus of which tends to infinity. Thus, $\psi_{\mathcal{G}}(z)$ has a root of multiplicity one at $1/p$ and no other roots.

Conversely, suppose $\psi_n(z)$ converges pointwise to a function with exactly one root at $z = 1/p$. Note that applying Cauchy's Integral Formula gives that the coefficients of $\psi_n(z)$ converge to the coefficients of the limit function $f(z) = \lim_{n\to\infty} \psi_n(z)$. Then f has the form

$$f(z) = \exp(bz + cz^2)(1 - pz)$$

by (1) and the fact that ψ_n is monic. Note that the coefficient of the linear term in this expression is $b - p$; that ψ_n has linear coefficient 0 implies that $b = p$. Thus, the z^2 coefficient of f is $c - p^2/2$. Since the z^2 coefficient of $\psi_n(z)$ is $-|E(G_n)|/n^2$, we have that $c = p^2/2 - \beta/2$ where $\beta = \lim_{n\to\infty} 2|E(G_n)|/n^2$ is the limiting edge density. This implies that the z^4 coefficient of f is $\beta^2/8 - p^4/4$. On the other hand, by Theorem 2, the z^4 coefficient of $\psi_n(z)$ is n^{-4} times the number $(\#2K_2 \subseteq G_n)$ of matchings of size 2 in G_n minus twice the number $(\#C_4 \subseteq G_n)$ of C_4 subgraphs. But, $(\#2K_2 \subseteq G_n) = \binom{|E(G)|}{2} = n^4\beta^2/8 + o(n^4)$, so

$$\frac{\beta^2}{8} - \frac{p^4}{4} = \frac{\beta^2}{8} - \lim_{n\to\infty} \frac{2(\#C_4 \subseteq G_n)}{n^4}$$

from which it follows that $t(C_4, G_n) + o(1) = |\operatorname{Aut}(C_4)| \cdot p^4/8 = p^4$ since $\operatorname{Aut}(C_4) \cong D_8$. On the other hand, the z^3 coefficient of $f(z)$ with $b = p$ and $c = p^2/2 - \beta/2$ is $-p^3/3$. The z^3 coefficient of $\psi_n(z)$ is $-2(\#C_3 \subseteq G_n)/n^3$, which implies that

$$\frac{p^3n^3}{6} + o(n^3) = (\#C_3 \subseteq G_n) \leq \frac{\beta^3 n^3}{6} + o(n^3)$$

since it is easy to see that a complete graph maximizes the number of triangles for a given number of edges. Then $\beta \geq p$ and so $t(C_4, G_n) \leq \beta^4 + o(1)$, implying the p-quasirandomness of the sequence G_n, by property P_2 in Theorem 1.

If G_n is 0-quasirandom, then G_n left-converges to the constant 0 graphon, whence $\psi_n(z) \to 1$ by Theorem 3 because all subgraph densities converge to 0 according to P_1. Conversely, if $\psi_n(z) \to 1$, then all coefficients converge to 0, so the edge density and C_4 density converge to zero, which gives 0-quasirandomness by P_2. □

3 Special Cases

Recall that $\operatorname{tr}\mathcal{G} = \sum_j \lambda_j$, the sum of the eigenvalues of (the kernel of) $\mathcal{G}$, and that $\mathcal{G}$ is "trace class" (aka "nuclear") if this sum converges absolutely. When $\mathcal{G}$ is trace class, we may write

$$\psi_{\mathcal{G}}(z) = \exp\left(\frac{(\|\mathcal{G}\|_2^2 - \|\mathcal{G}\|_1^1)z^2}{2} + z\operatorname{tr}\mathcal{G}\right)\prod_{s\in\mathcal{S}}(1 - sz), \tag{3}$$

a factorization of the characteristic power series into a monic polynomial-like product whose roots are the reciprocals of the nonzero eigenvalues of $\mathcal{G}$ and an exponential term. The quantity $\operatorname{tr}\mathcal{G}$ is zero if $\mathcal{G}$ is bipartite: in particular, the eigenvalues comes in pairs $\pm\lambda_j$. (For more on the spectra of bipartite graphs, see [9], in particular Theorem 8.) In this case, $\psi_G(z)$ has no monomials of odd degree:

$$\psi_{\mathcal{G}}(z) = \exp\left[(\|\mathcal{G}\|_2^2 - \|\mathcal{G}\|_1^1)z^2/2\right]\prod_{s\in\mathcal{S}\cap\mathbb{R}^+}\left(1 - s^2z^2\right),$$

Furthermore, $\|\mathcal{G}\|_2^2 = \|\mathcal{G}\|_1^1$ iff $\mathcal{G}$ is a 0-1 function except for a set of measure zero, as with a simple blow-up of a graph (sometimes called a "pixel diagram"), so the quadratic term vanishes in the exponential, resulting in

$$\psi_{\mathcal{G}}(z) = \exp\left(z\operatorname{tr}\mathcal{G}\right)\prod_{s\in\mathcal{S}}(1 - sz).$$

We can also use (3) to obtain a simple expression for the characteristic power series of p-quasi-random graphons.

Proposition 1. *G is p-quasirandom iff*

$$\psi_{\mathcal{G}}(z) = \sum_{k=0}^{\infty} z^k \sum_{\lambda\in\Lambda(k;i,j)} \frac{(-1)^j p^{k-i}}{\eta(\lambda)} = (1 - pz)\exp\left(pz - \frac{p(1-p)}{2}z^2\right) \tag{4}$$

where $\Lambda(k; i, j)$ is the set of integer partitions of k into j parts of size at least 2, of which i are of size exactly 2.

Proof. By Theorem 4, G is p-quasirandom iff

$$\begin{aligned}
\psi_{\mathcal{G}}(z) &= \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{H\in\mathcal{H}_k} z^k (-1)^{c(H)} \frac{(\#H \subseteq G)}{n^k \eta(H)} \\
&= \sum_{k=0}^{\infty} \sum_{H\in\mathcal{H}_k} z^k (-1)^{c(H)} \frac{p^{|E(H)|}}{\eta(H)} \\
&= \sum_{k=0}^{\infty} z^k \sum_{\lambda\in\Lambda(k;i,j)} \frac{(-1)^j p^{k-i}}{\eta(\lambda)}
\end{aligned}$$

because partitions of k into parts of size at least 2 correspond bijectively to elements of $\mathcal{H}_k$. That this equals the right-hand side of (4) follows from Theorem 4. However, we show the result directly here.

If $\alpha(z) = \exp(\beta(z))$ is a power series, where the z^n coefficient of α is a_k and the z^k coefficient of β is b_k, then (by standard facts about exponential generating functions, see, e.g., [4]), letting λ be an integer partition with m_i parts of distinct sizes c_i, $i = 1$ to t,

$$\begin{aligned}
a_k &= \frac{1}{k!} \sum_{\pi\in\Pi} \prod_{B\in\pi} b_{|B|} |B|! \\
&= \frac{1}{k!} \sum_{\lambda\vdash k} \frac{k!}{\eta(\lambda) \prod_{i=1}^{t} (c_i - 1)!^{m_i}} \prod_{i=1}^{t} b_i^{m_i} c_i!^{m_i} \\
&= \sum_{\lambda\vdash k} \frac{\prod_{i=1}^{t} b_i^{m_i} c_i^{m_i}}{\eta(\lambda)}
\end{aligned}$$

where Π is the set of (set) partitions of an k-set. Thus, if

$$\begin{aligned}
\beta(z) &= pz - z^2 p(1-p)/2 + \log(1-pz) \\
&= pz + \frac{z^2(p^2-p)}{2} - pz - \frac{(pz)^2}{2} - \frac{(pz)^3}{3} - \cdots \\
&= -z^2 \cdot \frac{p}{2} - z^3 \cdot \frac{p^3}{3} - z^4 \cdot \frac{p^4}{4} \cdots
\end{aligned}$$

and $\alpha(z) = \exp(\beta)$, then

$$\begin{aligned}
a_k &= \sum_{\lambda\vdash k} \frac{\prod_{i=1}^{t} b_i^{m_i} c_i^{m_i}}{\eta(\lambda)} \\
&= \sum_{\lambda\in\Lambda(k;r,s)} p^{-r} \frac{\prod_{i=1}^{t} (-1/c_i)^{m_i} c_i^{m_i} p^{m_i}}{\eta(\lambda)} \\
&= \sum_{\lambda\in\Lambda(k;r,s)} p^{-r} \frac{\prod_{i=1}^{t} (-p)^{m_i}}{\eta(\lambda)}
\end{aligned}$$

$$= \sum_{\lambda \in \Lambda(k;r,s)} (-1)^s p^{k-r} \frac{1}{\eta(\lambda)}.$$

□

4 Questions

It is tempting to define instead a characteristic power series without the regularization, i.e., $\det(I - z\mathcal{G}) = \prod_j (1 - \lambda_j z)$, the "Fredholm determinant". However, this product may not converge. Using Fourier series, it is straightforward to show that, for the graphon $\mathcal{G}(x, y)$ defined by

$$\mathcal{G}(x, y) = \begin{cases} 1 & \text{if } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\mathcal{G}(x, y) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi x}{2}\right) \cos\left(\frac{(2n+1)\pi y}{2}\right)$$

Since $\{f_n\}_{n=0}^{\infty}$ with $f_n(t) = 2\cos[(2n+1)\pi t/2]\}$ is an orthonormal family of functions on $[0, 1]$, this shows that the spectrum of $\mathcal{G}$ is $\left\{\frac{(-1)^n}{(2n+1)\pi}\right\}_{n=0}^{\infty}$. Since the (odd) harmonic series diverges, it follows that $\mathcal{G}$ is not trace-class, i.e., $\sum_i \lambda_i$ does not converge absolutely. However, it still converges conditionally, so $\det(I - z\mathcal{G})$ is well-defined.

One can also construct a function from $[0, 1] \times [0, 1] \to \mathbb{R}^2$ by setting $\mathcal{G}(x, y) = \sum_{n=0}^{\infty} \frac{1}{n} (-1)^{\epsilon_n(x)+\epsilon_n(y)}$ where $\epsilon_n(x)$ is the n-th digit of x in binary. Since $\{(-1)^{\epsilon_n}\}_{n=0}^{\infty}$ is the orthonormal Rademacher system, the spectrum of $\mathcal{G}$ is the harmonic series. Thus $\det(I - z\mathcal{G})$ is undefined, but because random harmonic series have full support on the real line, the function $\mathcal{G}$ is not a graphon (and cannot be linearly scaled to become one).

Thus, we are led to the following question.

Question 1. *Does there exist a graphon $\mathcal{G}$ for which the Fredholm determinant $\det(I - z\mathcal{G})$ does not exist?*

The characteristic polynomial of the p-quasirandom graphon has some possible connections with its probabilistic interpretations, as follows.

Question 2. *Let $\mathcal{G}$ be the p-quasirandom graphon. The function*

$$\psi_{\mathcal{G}}(z) = (1 - pz) \exp(pz - z^2 p(1-p)/2)$$

has some unexplained connections with Gaussian probability distributions. If $M(z)$ is the moment generating function of a normal distribution of mean p and variance $p(1-p)$ – the normalized limit of a 0-1 random walk with bias p – then $\psi_{\mathcal{G}}(z) = (1 - pz)/M(-z)$. Why?

Our next question concerns to what extent some of the above approach can be applied to the many other well-known graph polynomials: matching polynomial, Laplacian characteristic polynomial, chromatic polynomial, etc. When is it the case that, given some notion of graph convergence, such as left-convergence leading to graphons as above or Benjamini-Schramm convergence of very sparse graphs, these polynomials when suitably normalized converge to some power series? One motivation for asking this is a related, growing body of work on limits of measures supported on the roots of natural graph polynomials. For example, building off of work by Sokal [17] and Borgs-Chayes-Kahn-Lovász [5], Abért-Hubai [1] showed the convergence of harmonic moments (quantities $\int_K f\,d\nu$ for holomorphic functions f and certain regions $K \subset \mathbb{C}$) of the uniform probability distribution over the chromatic roots of Benjamini-Schramm-convergence graph sequences; subsequently, Csikvári-Frenkel [7] generalized this to a wide class of graph polynomials (including the characteristic polynomial) and Csikvári-Frenkel-Hladký-Hubai [8] showed that it holds even for dense graph (i.e., graphon) convergence with suitable normalization. From a different perspective, [19] empirically showed that chromatic roots of Erdős-Rényi random graphs appear to have a scaling limit.

Question 3. *For which other graph polynomials and graph limit process can the above type of analysis be carried out? How about for hypergraphs?*

Finally, we mention a question that arose in the context of experimentally computing the coefficients of the characteristic power series of that simplest of graphons, the uniform quasirandom graphon.

Question 4. *It is straightforward to show (by, for example, applying Turán's Inequalities; see [15]) that the coefficients c_k of $\psi_{\mathcal{G}}$ are log-concave for any graphon $\mathcal{G}$, but the consequences of this for unimodality are unclear because we do not know the sign pattern of the coefficients of $\psi_{\mathcal{G}}(z)$. For example, the characteristic power series of a $p = 1/2$ quasi-random graphon has sign pattern*

$$+, 0, -, +, +, -, -, +, +, +, -, -, +, +, -, -, \ldots$$

More specifically, can $c_k = 0$ if $k \neq 1$?

5 Acknowledgments

Thanks to G. Clark, C. Edgar, G. Fickes, V. Nikiforov, and A. Riasanovsky for helpful discussions. Thanks also to the anonymous referees for very useful suggestions, especially the identification of a need to use higher-order determinants.

References

[1] Miklós Abért and Tamás Hubai. Benjamini-Schramm convergence and the distribution of chromatic roots for sparse graphs. *Combinatorica*, 35(2):127–151, 2015.

[2] Andrew Bakan, Thomas Craven, and George Csordas. Interpolation and the Laguerre-Pólya class. *Southwest J. Pure Appl. Math.*, (1):38–53, 2001.

[3] Ralph Philip Boas, Jr. *Entire functions*. Academic Press Inc., New York, 1954.

[4] Miklós Bóna. *A walk through combinatorics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. An introduction to enumeration and graph theory, Fourth edition [of MR1936456], With a foreword by Richard Stanley.

[5] Christian Borgs, Jennifer Chayes, Jeff Kahn, and László Lovász. Left and right convergence of graphs with bounded degree. *Random Structures Algorithms*, 42(1):1–28, 2013.

[6] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[7] Péter Csikvári and Péter E. Frenkel. Benjamini-Schramm continuity of root moments of graph polynomials. *European J. Combin.*, 52(part B):302–320, 2016.

[8] Péter Csikvári, Péter E. Frenkel, Jan Hladký, and Tamás Hubai. Chromatic roots and limits of dense graphs. *Discrete Math.*, 340(5):1129–1135, 2017.

[9] Martin Doležal and Jan Hladký. Matching polytons. *Electron. J. Combin.*, 26(4):Paper No. 4.38, 33, 2019.

[10] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[11] Michael Griffin, Ken Ono, Larry Rolen, and Don Zagier. Jensen polynomials for the Riemann zeta function and other sequences. *Proc. Natl. Acad. Sci. USA*, 116(23):11103–11110, 2019.

[12] Frank Harary. The determinant of the adjacency matrix of a graph. *SIAM Rev.*, 4:202–210, 1962.

[13] László Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.

[14] Vladimir Nikiforov. Graphs and matrices with maximal energy. *J. Math. Anal. Appl.*, 327(1):735–738, 2007.

[15] J. Schur and G. Pólya. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. *J. Reine Angew. Math.*, 144:89–113, 1914.

[16] Barry Simon. Notes on infinite determinants of Hilbert space operators. *Advances in Math.*, 24(3):244–273, 1977.

[17] Alan D. Sokal. Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. *Combin. Probab. Comput.*, 10(1):41–77, 2001.

[18] Balázs Szegedy. Limits of kernel operators and the spectral regularity lemma. *European J. Combin.*, 32(7):1156–1167, 2011.

[19] Frank Van Bussel, Christoph Ehrlich, Denny Fliegner, Sebastian Stolzenberg, and Marc Timme. Chromatic polynomials of random graphs. *J. Phys. A*, 43(17):175002, 12, 2010.