Counting Antichains and Linear Extensions in Generalizations of the Boolean Lattice

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Abstract

In 1897, Dedekind posed the problem of estimating the number of antichains in the Boolean lattice; in particular, he asked whether the logarithm of the number is asymptotic to the size of the middle layer of the n-dimensional Boolean lattice. Dedekind's question is natural to ask for any ranked unimodal poset with the Sperner property, as the size of the largest layer is an upper bound on the size of the largest antichain. Further inspired by the elegant entropy approaches for Dedekind's original problem by Pippenger and Kahn, in the present work we consider enumeration of antichains in two natural generalizations of the Boolean lattice: the chain product poset and the poset of partially defined functions. We also consider the related problem of counting the number of linear extensions of these two classes of posets. Our main contribution is that of obtaining accurate (up to first order) asymptotics of the logarithm of the number of antichains and of the number of linear extensions.

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1 Introduction

In 1897, Dedekind [4] had posed the problem of estimating the number of antichains in the Boolean lattice; in particular, he had asked whether the logarithm of the number is asymptotic to $\binom{n}{\lfloor n/2 \rfloor}$, the size of the middle layer of the n-dimensional Boolean lattice $\mathfrak{B}_n$. Since the resolution of the problem in the affirmative by Kleitman [8] in 1969, much research has been devoted to investigating this problem ([9], [12], [7], ...), culminating in the works of Korshunov [10] and Sapozhenko [14], who derived sharp estimates on the actual (rather than the logarithm of the) number of linear extensions.

Dedekind's question is natural to ask for any ranked unimodal poset with the Sperner property, as the size of the largest layer is an upper bound on the size of the largest antichain. Further inspired by the elegant entropy approaches for Dedekind's original problem by Pippenger [12] and Kahn [7], in the present work we consider enumeration of antichains in two natural generalizations of the Boolean lattice: the chain product poset denoted by $[t]^n$, and the poset of partially defined functions, $F_{n,k}$; which will both be defined precisely in the following. Our first main contribution is that of obtaining accurate (up to first order) asymptotics of the logarithm of the number of antichains of both these posets.

The problem of counting the number of linear extensions of the Boolean Lattice has been raised by Stanley (see [15]) and others independently. Before describing our second main contribution, we recall (as motivation) the fact that Brightwell-Tetali [2] (improving on an earlier result of Sha-Kleitman [15]) obtained accurate (up to *second order*) asymptotics for the logarithm of the number of linear extensions of $\mathfrak{B}_n$. Once again, as a natural extension, we consider the analogous questions for the generalized Boolean lattices, $[t]^n$ and $F_{n,k}$. Once again, we provide the first results in this direction by providing accurate first order term asymptotics for the logarithm of the number of linear extensions of these posets.

We hope our work inspires more research in this direction in ultimately yielding much sharper estimates. In the next section, we state our results in precise terms, while providing some basic information on the posets under consideration.

1.1 Statement of Results

We begin by recalling the definition of a standard generalization of the Boolean lattice.

Definition 1.1 (The chain product poset $[t]^n$). *Each element in $[t]^n$ is represented by an n-tuple, $(x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq t-1$ for all $i = 1, \ldots, n$. For $y = (y_1, y_2, \ldots, y_n)$, we say $x \prec y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$.*

Notice that the Boolean lattice $\mathfrak{B}_n$ is isomorphic to $[2]^n$. This is a ranked poset, as we can assign $\text{rank}(x) = \sum_{i=1}^n x_i$.

Let $N(t,n)$ be the size of the middle layer, which corresponds to rank $n(t-1)/2$, of the chain product $[t]^n$. It was recently shown by Mattner and Roos [11] that

$$N(t,n) = t^n \sqrt{\frac{6}{\pi(t^2-1)n}}\,(1+o(1))\,. \tag{1}$$

Let $a([t]^n)$ and $L(F_{n,k})$ denote the number of antichains and the number of linear extensions of $[t]^n$, respectively. Then most of our technical contribution is towards the following theorem.

Theorem 1.2 *For integers t, n such that $1 < t < n$:*

$$N(t,n) \le \log(a([t]^n)) \le N(t,n)\left(1 + \frac{11t^2 \log t\ (\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right),$$

$$(n-1)\ln t - \ln n - 1 \le \frac{\ln L([t]^n)}{t^n} \le (n-1)\ln t + O(n^{1/4}\sqrt{\ln t})\,.$$

Note that in the first equation of the theorem, the lower bound is obvious, while the additional term in the upper bound is $o(1)$, as long as $t = O(n^{1/8-\epsilon})$, for any $\epsilon > 0$.

We also prove estimates, accurate to first order, for the analogous enumeration problems on the following poset, which is a different generalization of the Boolean lattice.

Definition 1.3 (The poset of partially defined functions $F_{n,k}$). *For $n, k \in \mathbb{N}$ with $k \ge 1$, let $F_{n,k}$ represent the poset of partially defined functions from $[n]$ to $[k]$ ordered by inclusion.*

We will use $\mathcal{D}(f)$ to denote the domain of a function f. Functions f and g satisfy that $f \preceq g$ if and only if $\mathcal{D}(f) \subseteq \mathcal{D}(g)$ and $f(x) = g(x)$, for all $x \in \mathcal{D}(f)$. In other words, $f \preceq g$ in $F_{n,k}$ if and only if g is an extension of f.

It is easy to see that for every integer $n \ge 1$, the poset $F_{n,1}$ is isomorphic to the Boolean lattice B_n, with $|F_{n,k}| = (k+1)^n$. (We see that each element in $F_{n,1}$ can be thought of as a string using the alphabet $\{0,1\}$, where a 0 represents the undefined character.) It is also easy to check that $F_{n,k}$ is a ranked poset on $n+1$ levels; letting P_i denote the set of elements of rank i, we have $|P_i| = k^i\binom{n}{i}$, with the mode being at $i = i_{\max} = \left\lfloor \frac{kn}{k+1} \right\rfloor$. Thus we have

$$|P_{i_{\max}}| = k^{\left\lfloor \frac{kn}{k+1} \right\rfloor} \binom{n}{\left\lfloor \frac{kn}{k+1} \right\rfloor}\,. \tag{2}$$

Also, let

$$\mathcal{L}_{n,k} = \frac{1}{(k+1)^n} \log \left(\prod_{i=0}^{n} \left(\binom{n}{i} k^i \right)! \right).$$

As before, let $a(F_{n,k})$ and $L(F_{n,k})$ denote the number of antichains and the number of linear extensions of $F_{n,k}$, respectively.

Theorem 1.4 *For positive integers k and n, with $i_{\max}$ and $P_{\mathrm{i}_{\max}}$ defined as above, we have:*

$$|P_{i_{\max}}| \ \le \log(a(F_{n,k})) \le \ |P_{i_{\max}}| \left(1 + O\Big(\frac{\log\{i_{\max}\}}{i_{\max}}\Big)\right),$$

$$\mathcal{L}_{n,k} \ \le \ \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \le \mathcal{L}_{n,k} + O\Big(\frac{\log n}{n}\Big).$$

The lower estimates in the above theorem are obvious bounds. On the other hand, we adopt a proof technique of Pippenger [12] to derive the first upper bound, whereas the second upper bound in the theorem is a straightforward corollary of a general upper bound, on the number of linear extensions of any ranked poset with degree regularity, proved in [2].

1.2 Further Background

It is worth remarking here that both the generalizations we consider here lack some of the 'nice' properties of the Boolean lattice, such as symmetry about a central rank and a single parameter which determines the poset. Additionally, $[t]^n$ lacks degree regularity within level sets and $F_{n,k}$ lacks a lattice structure, as it has no unique maximal element. These structural differences impede use of the techniques of Saphozhenko [14] and Korshunov [10], which give asymptotics for the number of antichains itself. We now provide a bit more information on the structure of these posets, which we use in the later sections.

First observe that $[t]^n$ is a ranked poset; however, it is not bi-regular, as the degree of a vertex does not depend only on its rank. The up degree of a vertex is equal to the number of positions where it takes a value less than n, and the down degree is equal to its number of nonzero entries.

$F_{n,2}$ is isomorphic to the dual of the cubic poset, Q_n, which is the poset of lower dimensional faces of B_n (not including the empty set $\emptyset$) ordered by inclusion. We can think of Q_n as the set of n-tuples taking values in 0,1,2 ordered by $\vec{b} \le \vec{c}$ if and only if $c_i = 2$ or $b_i = c_i$ for all $i \in [n]$. Given a set $I \in [n]$, for $i \in I$ we fix values $a_i = \alpha_i \in \{0, 1\}$. If we then let a_i range over 0 and 1 for all $i \notin I$, this determines a face of B_n.

Indexing the $n + 1$ levels of $F_{n,k}$ by the values $i \in \{0, \ldots, n\}$, the set of functions with exactly i values defined forms the i^{th} level, A_i. The size of the A_i is $n_i = k^i\binom{n}{i}$. The levels are indexed so that levels closer to the "bottom" of the poset have lower indices. As mentioned earlier, it is easy to check that the rank sequence $\{r_i\}_{i=0}^{n}$ is unimodal, with the mode occurring at rank $\left\lfloor \frac{kn}{k+1} \right\rfloor$.

Similar to $\mathfrak{B}_n$, $F_{n,k}$ is also a graded poset: there is a unique minimum element, and all maximal elements have the same rank. Additionally, the up and down degrees of an element are determined by the level it belongs to. If f is a member of level i it is defined in exactly i coordinates, with the remaining $n-i$ coordinates left undefined. The down degree d_i of level i is equal to i for each level. To select a downward neighbor of f, we just need to pick any of the i coordinates where it is undefined and replace the value there with the undefined character. The up degree of level i for $i = 0, \ldots, n-1$ is $u_i = k(n-i)$: to find an upward neighbor of f, we first choose one of the $n-i$ coordinates which is undefined in f and assign it one of the k allowed values.

2 Dedekind's Problem on Generalized Boolean Lattices

2.1 Number of Antichains in $[t]^n$

An easy lower bound for the number of antichains contained in $[t]^n$, utilizes the quantity N from Equation (1), as each subset of the middle layer is in itself an antichain. We will show, using an information theoretic technique, that 2^N asymptotically approximates the number of antichains in $[t]^n$. More precisely,

Theorem 2.1 *Let $a([t]^n)$ be the number of antichains in $[t]^n$, then for $n \geq 4$, $t = o(n^\epsilon)$ with $0 < \epsilon \leq \frac{1}{8}$, we can say that:*

$$N \leq \log(a([t]^n)) \leq N\left(1 + \frac{11t^2 \log t \ (\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right).$$

The $t = o(n^\epsilon)$, $0 < \epsilon < \frac{1}{8}$ hypothesis is necessary to ensure that this theorem gives matching first order terms at the logarithmic level. In the proof itself we use a weaker hypothesis on the relationship of t and n– the theorem remains true when $t = \omega(n^{\frac{1}{8}})$, however in this case the result gives us no useful information.

We follow closely a method used by Pippenger [12], which he used to show a similar result for the number of antichains contained in the Boolean lattice:

Theorem 2.2 *Let $a(\mathfrak{B}_n)$ be the number of antichains in the Boolean lattice. Then:*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \log(a(\mathfrak{B}_n)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right). \tag{3}$$

We know that Pippenger's result does not give the best known asymptotics for the number of antichains in the Boolean Lattice. In particular, the second order term is far from what we know to be the truth in this case, which was discovered by Korshunov [10], who developed asymptotics for the function $a(\mathfrak{B}_n)$ directly. The detailed case analysis involved in his argument is very precise, and difficult to reproduce in a ranked poset which is not bi-regular. A result of this strength is seemingly out of reach with current entropy techniques. We intend Theorem 2.1 as a preliminary result, to establish that N gives the correct first order term at the logarithmic level for the number of antichains in $[t]^n$.

For our information theoretic approach, we will use two functions extensively, $H(X)$, the entropy of a random variable (see [1] for information on entropy), and $h_1(p)$, the truncated binary entropy function. We define

$$h_1(q) = \begin{cases} -q\log_2 q - (1-q)\log_2(1-q) & \text{if } 0 \leq q \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq q \leq 1. \end{cases} \tag{4}$$

For the remainder of the paper, we will use log for $\log_2$. A function f defined on a poset $(P, \preceq)$ is monotone if $x \preceq y$ implies that $f(x) \leq f(y)$. We note that there is a one-to-one correspondence between antichains and monotone Boolean functions in a ranked poset. We can represent an antichain, $\mathcal{A}$, with a Boolean function g, by taking $g(a) = 1$ for all $a \in \mathcal{A}$, and $g(b) = 0$ otherwise. From each such function g, we can form the associated monotone Boolean function f, by "closing g upwards," i.e. by setting $f(z) = 1$ if $x \preceq z$ for some x for which $g(x) = 1$. Similarly, given a monotone Boolean function f, note that the set of minimal elements x, for which $f(x) = 1$, defines an antichain. This correspondence allows us to count antichains by counting monotone (Boolean) functions in $[t]^n$.

We will use the following lemma from [12] in the course of our argument.

Lemma 2.3 *Suppose the random variable K takes values in $\{0, 1, \ldots, n\}$, and for some $k \geq 1$ and $0 \leq q \leq 1$,*

$$\mathbb{P}(K \geq k) \leq q,$$

Then $H(K) \leq h_1(q) + \log k + q \log n$.

Note that $[t]^n$ is a Sperner poset, so the largest antichain in $[t]^n$ is a level set. This follows from the fact that it is the product of chains, which are Sperner. Together with Dilworth's Theorem, which states that there is a chain partition whose size is equal to the size of the largest antichain, we are guaranteed that there is a chain partition of $[t]^n$ whose size is exactly N. (See [5] for additional details on Sperner posets and chain partitions). We will fix one such chain partition for the remainder of our argument. We will enumerate the elements of this chain partition with $\{C_1, C_2, \ldots, C_N\}$. For a fixed monotone function, g, there is a unique point in each chain where the values of the function change from zero to one. For each chain C_j in the partition, we can define a parameter to capture this information,

$$\gamma_j(g) = \#\{x \in C_j : g(x) = 1\}\,.$$

Our strategy for counting the number of monotone functions will hinge on this property. Let f be a monotone Boolean function, chosen uniformly from the set of all monotone Boolean functions. We know from a basic property of entropy that

$$H(f) = \log(a([t]^n)).$$

We will define a pair of variables $(\hat{\delta}, \tilde{\delta})$, which will in turn determine f. This will allow us to use the subadditivity of entropy again to give an upper bound on $H(f)$:

$$H(f) \leq H(\hat{\delta}, \tilde{\delta}) \leq H(\hat{\delta}) + H(\tilde{\delta}). \tag{5}$$

To define $\tilde{\delta}$ we first need another description. For each point $x \in [t]^n$, and $\ell = 0, \ldots, t-1$, define $d_\ell(x)$ to be the number of coordinates of x which take value ℓ. Let $d_{\text{down}}(x)$ be the down degree of x, the number of neighbors of x on the immediately preceding level. We can think of $d_\ell(x)$ as the ℓ^{th} down degree of x, as $\sum_{\ell=1}^{t-1} d_\ell(x) = d_{\text{down}}(x)$. We will call a point in $[t]^n$ *low* if $d_j(x) < \frac{n}{2t}$ for some $1 \leq j \leq t-1$. The word low is a vestigial artifact of Pippenger's proof, where he called points low if $d_1(x) \leq \frac{n}{4}$. These points directly correspond to all points in the Boolean lattice which occur on the lowest $\frac{n}{4}$ levels. In the context that we use the word low, it is important to note that it is possible to have two points in the same level set where one is low and the other is not, so low is not purely rank dependent. *This is a new ingredient of our proof.* We will call a chain low if it contains a low point.

We will take f and selectively "forget" information from some of the chains in the chain partition. To be more precise, let $v_1, v_2, \ldots, v_N$ be independent random variables assigned to each chain in the chain partition. Let $p = \dfrac{(\log n)^{\frac{1}{2}}}{n^{\frac{1}{4}}}$. With each chain C_j, we associate a variable as follows:

$$v_j = \begin{cases} 1 \text{ if } C_j \text{ is low} \\ 1 \text{ with probability } p \text{ for } C_j \text{ not low} \\ 0 \text{ with probability } 1-p \text{ for } C_j \text{ not low} \end{cases} \tag{6}$$

From f we form the function $\tilde{\delta} = (\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_N)$, by taking $\delta_j = \gamma_j(f)v_j$. Now $\tilde{\delta}$ gives us enough information to reconstruct f on all low chains and on any chain C_j, with $v_j = 1$. Let $\tilde{f}$ be the smallest monotone function which is consistent with $\tilde{\delta}$, i.e. the smallest function so that $\gamma_j(\tilde{f}) \geq \tilde{\delta}_j$, for all $1 \leq j \leq N$. We record the missing information about f in a variable $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_N)$ where $\hat{\delta}_j = \gamma_j(f) - \gamma_j(\tilde{f})$. From these definitions, it is easy to see that f is determined by $\delta = (\tilde{\delta}, \hat{\delta})$; indeed, $\tilde{\delta}$ contains the information about g on all chains which are not forgotten, and we can reclaim information about the rest of the chains using the information from $\hat{\delta}$.

We now want to bound $H(\tilde{\delta})$. First we use the subadditivity of entropy to say that:

$$H(\tilde{\delta}) \leq \sum_{j=1}^{N} H(\tilde{\delta}_j). \tag{7}$$

For each fixed j, we now bound $H(\tilde{\delta}_j)$ using Lemma 2.3. Observe that $\tilde{\delta}_j \geq 1$ only if $v_j = 1$. We can use $\mathbb{P}(v_j = 1)$ in place of q in the lemma, so that $\mathbb{P}(\tilde{\delta}_j \geq 1) \leq \mathbb{P}(v_j = 1)$ implies that $H(\tilde{\delta}_j) \leq h_1(q) + q \log n$. If C_j is low then $v_j = 1$ with probability 1. Then $\tilde{\delta}_j = \gamma_j$, a random variable taking either value 0 or a value in $\{1, \ldots, n\}$, since this records how many ones are in our low chain. Therefore if C_j is low, $H(\tilde{\delta}_j) \leq 1 + \log(n)$. If C_j is not low, we can use Lemma 2.3 with $q = p$. Letting M be the number of low chains, we can separate the terms of the sum as follows:

$$H(\tilde{\delta}) \leq M(1 + \log n) + (N - M)(h_1(p) + p \log n). \tag{8}$$

To proceed, we need to give a bound on M, the number of low chains. We can bound this from above by the number of low points. In order for a point x to be low, we need there to be some j for which $d_j(x) \leq \frac{n}{2t}$. We can think of each coordinate x_i, $i = 1, \ldots, n$, as a uniform random variable chosen from $\{0, 1, \ldots, t-1\}$. Then $\mathbb{P}(x_i = j) = \frac{1}{t}$, so the expected value for $d_j(x) = \frac{n}{t}$. Using a version of Chernoff's inequality from [6], we see that $\mathbb{P}(d_j(x) \leq \frac{n}{2t}) \leq \exp(-\frac{(\frac{n}{2t})^2}{2\frac{n}{t}})$. Since this is for a single value of j, we multiply by both t and the total number of points, to see that there are at most $t^{n+1} \exp(-\frac{n}{8t})$ low points.

Plugging this estimate and the value for p back into (8) and recalling the value of N from (1), we see that:

$$\begin{aligned} H(\tilde{\delta}) &\leq t^{n+1} e^{-\frac{n}{8t}} (1 + \log n) + N \left(\frac{3(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} \right) \\ &\leq N \left(\frac{4(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} \right). \end{aligned} \tag{9}$$

In the first inequality above, we use that for p small, $h_1(p) \le 2p\log(1/p)$. In the second inequality we use the fact that $t = o(n^\epsilon)$ (Though for this calculation it suffices for $t \le \frac{n}{16\log n}$).

Now we need to bound $H(\hat{\delta})$. Working with $\hat{\delta}$, we want to explore the possible discrepancy between f and $\tilde{f}$. We will call a chain C_j bad if $\hat{\delta}_j \ge 2$. Again, the first property we use is the subadditivity of entropy to say that:

$$H(\hat{\delta}) \le \sum_{i=1}^{N} H(\hat{\delta}_i).$$

We can again apply Lemma 2.3, now allowing $k = 2$ and setting $q_j = \mathbb{P}(\hat{\delta}_j \ge 2)$. Let $Q = \sum_{j=1}^{N} q_j$, noting that this is the expected number of bad chains. Then we can continue to say that:

$$\begin{aligned} H(\hat{\delta}) &\le \sum_{j=1}^{N}(h_1(q_j) + 1 + q_j \log n) \\ &\le N h_1\left(\frac{Q}{N}\right) + N + Q\log n \\ &= N\left(1 + h_1\left(\frac{Q}{N}\right) + \frac{Q}{N}\log n\right). \end{aligned} \tag{10}$$

where we use the concavity of entropy in the second inequality.

We will break into two cases to find a bound on Q. A point x in $[t]^n$ is called bad if (i) x is not low, (ii) the chain C_j which contains x also contains some y so that y is a neighbor of x on the immediately preceding level with $f(y) = 1$, and (iii) $\tilde{f}(x) = 0$. Each one of x's immediately preceding neighbors arises by decreasing one of the nonzero coordinates of x by one. If y differs from x in a coordinate where the value of x is k, we will refer to y as a k-neighbor of x. We classify bad points into two groups, points which are bad because they have many such k-neighbors for some $k \ge 1$, and points which have relatively few k-neighbors for some $1 \le k \le t-1$. To formalize this, let $s = n^{\frac{1}{4}}(\log n)^{\frac{1}{2}}$. We will call a point x heavy if for any k, x has more than s k-neighbors. In order for a heavy x to be bad, we need each of the (at least s) chains containing k-neighbors y w ith $f(y) = 1$ to be assigned a γ value of 0. This happens with probability $(1-p)$ for each chain. Using $p = \dfrac{(\log n)^{\frac{1}{2}}}{n^{\frac{1}{4}}}$ allows the following calculation:

$$\mathbb{P}(x \text{ is heavy and bad}) \le (t-1)(1-p)^s \le (t-1)e^{-ps} \le \frac{t}{n}.$$

The factor of $(t-1)$ exists because there are $t-1$ different k values for which x can be heavy. If x is not heavy, it means that for every k, the number of k-neighbors of x is less than s. We apply the group Sym(n), the group of all permutations on $[n]$. The subgroup Stab(x), which fixes x acts transitively on each collection of k-neighbors. Let y be a k-neighbor of x so that $f(y) = 1$. We can average over the whole orbit of y, since x is only bad if the chain containing it also contains y. Then the probability that x is bad but not heavy is bounded by $\frac{s}{\frac{n}{2t}} = \frac{2ts}{n} = \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}$.

Combining these two estimates, we see that for $n \geq 4$,

$$\mathbb{P}(x \text{ is bad}) \leq \max\left(\frac{t}{n}, \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}\right) = \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}.$$

Therefore $Q = \mathbb{E}(\text{bad points}) \leq t^n \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}$. We use bounds for Q and N in (10) to see that:

$$\begin{aligned} H(\hat{\delta}) &\leq N\left(1 + h_1\left(\frac{Q}{N}\right) + \frac{Q}{N}\log n\right) \\ &\leq N + \left(\sqrt{\frac{\pi}{24}}\right)\frac{t^2(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} + \left(2\sqrt{\frac{\pi}{6}}\right)\frac{t^2(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} \\ &\leq N + \frac{7t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}}. \end{aligned} \tag{11}$$

Now we are ready to give estimates for both parts of (5), to finally conclude that:

$$\begin{aligned} H(f) &\leq H(\tilde{\delta}) + H(\hat{\delta}) \\ &\leq N + \frac{7t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}} + N\left(\frac{4(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right) \\ &\leq N\left(1 + \frac{11t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}}\right), \end{aligned} \tag{12}$$

as claimed.

2.2 Number of Antichains in $F_{n,k}$

In this section, our goal is to give an estimate for $\log(a(F_{n,k}))$. The following theorem of Kahn [7], will play a key role:

Theorem 2.4 (Kahn) *Let P be a graded poset with levels $P_1, P_2, \ldots, P_m$, with $|P_m| \leq M$. Assume that there exists an $s \in \mathbb{N}$ (so that s is a uniform bound), with $s \geq d_{up}(v)$ and $s \leq d_{down}(v)$ for all $v \in P$, then*

$$a(P) \leq (m2^s - (m-1))^{\frac{M}{s}}. \tag{13}$$

We note that Kahn proved this theorem so he could bound the number of antichains in the Boolean Lattice $B_n \cong F_{n,1}$. However, the theorem cannot be applied directly, as neither $F_{n,k}$ nor B_n satisfy all of the hypothesis of the theorem. Kahn proves a technical lemma allowing him to apply his theorem indirectly to B_n; we will prove a similar technical lemma which allows us to extend his result to $F_{n,k}$.

The i-th level set in $F_{n,k}$ has size $|P_i| = \binom{n}{i}k^i$, for $i = 0, \ldots, n$. Let $s_{n,k}$ be the index of the largest level set in $F_{n,k}$. By maximizing this quantity in terms of i, we see that $s_{n,k} = \left\lfloor \frac{kn}{(k+1)} \right\rfloor$. Note that when $k = 1$, as in the case of $\mathfrak{B}_n$, we reclaim the well known fact that the largest level of the Boolean lattice has rank $\lfloor \frac{n}{2} \rfloor$. This definition leads directly to the lower bound in the following theorem, as any subset of the largest level set is itself an antichain.

Theorem 2.5 $2^{|P_{s_{n,k}}|} \leq a(F_{n,k}) \leq \left(s_{n,k}2^{s_{n,k}} - s_{n,k} - 1\right)^{\frac{|P_{s_{n,k}}|}{s_{n,k}}}$.

Calculating the quantity $\dfrac{\log(a(F_{n,k}))}{|P_{s_{n,k}}|}$ gives us a way of seeing how close the bounds are. As we can see, this result gives matching first order terms at the logarithmic level.

Corollary 2.6 *Recalling that* $s_{n,k} = \lfloor \frac{kn}{k+1} \rfloor$*,*

$$1 \leq \frac{\log(a(F_{n,k}))}{|P_{s_{n,k}}|} \leq 1 + O\left(\frac{\log(s_{n,k})}{s_{n,k}}\right).$$

$F_{n,k}$ does not satisfy the hypothesis for Kahn's theorem, as its rank-sequence is unimodal, and there is no uniform bound on degrees which applies to all levels. However, if we truncate $F_{n,k}$ at level $s_{n,k}$, the mode of the rank sequence, the truncated poset has a strictly increasing rank sequence and it satisfies that $d_{\text{up}}(x) \geq s_{n,k}$ for all $x \in P_1 \cup P_2 \ldots \cup P_{s_{n,k}-1}$ and $d_{\text{down}}(x) \leq s_{n,k}$ for all $x \in P_2 \cup P_3 \ldots \cup P_{s_{n,k}}$. Note that for $x \in P_i$, $d_{\text{up}}(x) = k(n-i)$ and $d_{\text{down}}(x) = i$; so $s_{n,k}$ appropriately plays the role of s in Theorem 2.4. Since we can use Kahn's theorem to count the number of antichains in the truncated poset, we seek a way to relate that number to the number of antichains in all of $F_{n,k}$.

Definition 2.7 *A poset Q is a* ***relaxation*** *of a poset P, if P and Q have the same groundset and $y \lessdot x \in Q \Rightarrow y \lessdot x \in P$.*

It is a simple consequence of the definition of relaxation that if Q is a relaxation of P, then $a(Q) \geq a(P)$. The Hasse diagram of P may contain more edges than that of Q, but adding edges only decreases the number of possible antichains. In order to establish the upper bound in Theorem 2.5, we seek a poset which satisfies the hypothesis of Kahn's theorem and

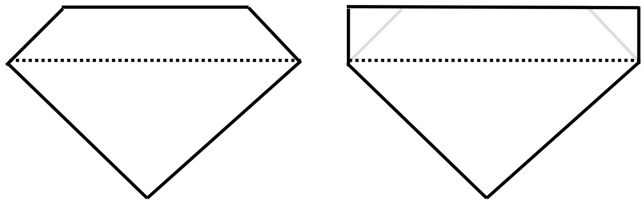

Figure 1: Heuristic drawings showing the shape of $F_{n,k}$ and $A_{n,k}$

contains a relaxation of $F_{n,k}$ as a subposet. Kahn has already constructed such a poset for $F_{n,1} \cong B_n$, and here we prove the following technical lemma for $k \geq 2$:

Lemma 2.8 *Fix n and $k \in \mathbb{N}$ with $k \geq 2$; there exists a graded poset $A_{n,k}$ ranked by $\{0, 1, \ldots, n\}$ which:*

- *Contains a relaxation of $F_{n,k}$;*
- *Satisfies $d_{up}(x) \geq s_{n,k}$ for all $x \in A_1 \cup A_2, \ldots \cup A_{n-1}$ and $d_{down}(x) \leq s_{n,k}$ for all $x \in A_2 \cup A_3, \ldots \cup A_n$;*
- *Satisfies $|A_i| \leq |P_{s_{n,k}}| = \binom{n}{s_{n,k}} (k^{s_{n,k}})$ for all level sets $\{A_i\}_{i=0}^{n}$.*

Proof. Take the Hasse diagram for $F_{n,k}$, with $k \geq 2$, and consider it for the moment as a graph. Our objective is to manipulate it to form the Hasse diagram of an appropriate poset $A_{n,k}$, and then take the transitive closure of the covering relations represented as edges of this Hasse diagram to form the desired poset. We do not need to make any modifications below level $s_{n,k}$, as these vertices already satisfy the degree bounds, and the ranks are increasing up to level $s_{n,k}$. We follow three steps to transform the Hasse diagram, and then justify that these steps can indeed be carried out.

1. for $i > s_{n,k}$, remove $i - s_{n,k}$ down edges from each vertex on level i. These edges can be chosen arbitrarily. This gives us $d_{\text{down}}(y) = s_{n,k}$ for every vertex $y \in P_{s_{n,k}} \cup \ldots \cup P_n$.
2. for $i > s_{n,k}$, add vertices so there are $k^{s_{n,k}}\binom{n}{s_{n,k}}$ vertices on level i. Let us call the number of vertices needed on level i, j_i, and note that $j_i = k^{s_{n,k}}\binom{n}{s_{n,k}} - k^i\binom{k}{i}$. This insures that the rank sequence of $A_{n,k}$ will be increasing.
3. for $i \geq s_{n,k}$, add edges between level i and $i+1$ to ensure that both the up and down degrees of every vertex above level $s_{n,k}$ are exactly equal to $s_{n,k}$, while maintaining that $F_{n,k}$ is contained as a relaxation.

The feasibility of the first two steps is clear. However in the third step, we need to verify that we can add enough edges to satisfy the degree requirements without adding additional edges between vertices in the original $F_{n,k}$ ground set to guarantee that our new graph will contain a relaxation of $F_{n,k}$.

After the first step, counting edges by their top endpoint, we see that there are $s_{n,k}\left(k^{i+1}\binom{n}{i+1}\right)$ edges remaining between levels i and $i+1$. Since we know vertices in level i require up degree $s_{n,k}$, we can see that the number of edges which we need to add between level i and $i+1$ is $s_{n,k}\left(k^i\binom{n}{i}\right) - s_{n,k}\left(k^{i+1}\binom{n}{i+1}\right)$. We need to be able to add all of these edges between vertices in $F_{n,k}$ on level i and new vertices on level $i+1$ in order to increase the up degree for vertices in level i to at least $s_{n,k}$.

Since the down degree of new vertices on level $i+1$ needs to be $s_{n,k}$, we can use all of the available edges from $L = k^i\binom{n}{i} - k^{i+1}\binom{n}{i+1}$ new level $i+1$ vertices to supplement the up degree of original vertices from level i. This leaves $k^{s_{n,k}}\binom{n}{s_{n,k}} - k^i\binom{n}{i}$ vertices on level $i+1$ which currently have degree 0 (Notice that this is exactly the number of added vertices on level i). At this point, all vertices have the correct degree except these new vertices which remain isolated on level $i+1$ and all of the added vertices on level i which have not been modified in the above procedure (all of them still have up degree 0 at this point).

We can label these isolated vertices in levels i and $i+1$ with labels $1 \dots j_i$ and $1 \dots j_{i+1} - L$ respectively. Now we add edges so that every vertex x_l on the bottom gets connected with each vertex on the top y_k for $k \in \{l, l+1, \dots, l+s_{n,k}\}$ where sums are evaluated modulo j_i. Since $s_{n,k} \leq j_i$ this process creates a simple graph, a the circulant bipartite graph with degree $s_{n,k}$.

This now determines the complete Hasse diagram for $A_{n,k}$ by giving all of its covering relations. Taking the transitive closure of these relations gives us a poset satisfying the hypothesis of Theorem 2.4. □

Proof of Theorem 2.5 follows by applying Theorem 2.4 on $A_{n,k}$ to give an upper bound for $a(A_{n,k})$, noting that $a(F_{n,k}) \leq a(A_{n,k})$ since we ensured that it contained a relaxation of $F_{n,k}$. We use the values $m = s_{n,k}$ and $M = k^{s_{n,k}}\binom{n}{s_{n,k}}$.

3 Counting Linear Extensions

Definition 3.1 *Given a poset $(P, \preceq)$, a* ***linear extension*** *of P is a total ordering $(T, \leq)$ on the ground set of P which preserves the relationship $\preceq$. In other words if $x \preceq y$ in P, then $x \leq y$ in T.*

Let $L(P)$ denote the number of linear extensions of a poset P.

Proposition 3.2 *Let P be a ranked poset with height K. If we enumerate the levels of P as P_i for $i = 1, \ldots, K$, with associated rank sequence $\{r_i\}_{i=1}^K$, a lower bound for $L(P)$ is given by the expression*

$$\prod_{i=1}^{K} r_i! \le L(P)\,. \tag{14}$$

Proof. The right hand side counts the number of linear extensions which can be formed by ordering each level set individually and then concatenating the orderings of each level, putting all elements of a given level before all of the elements in a higher level.

The first non-trivial upper bound for $L(P)$ for ranked posets was given by Sha and Kleitman [15]. They proved that:

Theorem 3.3 *(Sha and Kleitman) For a ranked poset, P, as above:*

$$L(P) \le \prod_{i=1}^{K} (r_i)^{r_i}.$$

In the next section, we will use this result to get an upper bound on the number of linear extensions of the chain product poset $[t]^n$.

To derive an upper bound on the number of linear extensions of $F_{n,k}$, the other poset under consideration in this work, we use a theorem of of Brightwell-Tetali [2]. In [2], the main focus was on deriving a tight bound (correct up to a second order term) on the logarithm of the number of linear extensions of the Boolean lattice. In addition to this, the authors gave a general upper bound for the number of linear extensions of posets with degree-regularity: Let u_j be the up degree for all vertices on level j, and similarly let d_j be the down degree of all vertices on level j. Letting r be the following harmonic average of the down degrees:

$$r = \sum_{j=2}^{K} \frac{r_{j-1}}{d_j}\,, \tag{15}$$

they showed:

Theorem 3.4 *(Brightwell and Tetali) For a regular poset, P with $|P| = N$ and height $K > 0$, as above:*

$$L(P) \le \prod_{i=1}^{K} r_i! \left(\frac{2e(K-1)N}{r} \right)^r. \tag{16}$$

3.1 Linear Extensions of $[t]^n$

First, we use convexity and standard estimates to give a lower bound for $L([t]^n)$.

Proposition 3.5

$$L([t]^n) \geq \left(\frac{t^{n-1}}{ne}\right)^{t^n}.$$

Proof. By Proposition 3.2, a lower bound for the number of extensions of any ranked poset is the product of the factorials of its rank sequence. Let $r_0, \ldots, r_{n(t-1)}$ denote the rank sequence of the poset $[t]^n$, i.e., $r_j = |P_j|$. We claim that, for any sequence of positive integers $b_1, \ldots, b_s$,

$$\prod_{j=1}^{s} b_j! \geq \left(\left(s^{-1}\sum_{j=1}^{s} b_j\right)!\right)^{s}, \tag{17}$$

where $x! = \Gamma(x+1)$ for $x \in \mathbb{R}$. To see this, take the natural log of both sides:

$$\sum_{j=1}^{s} \ln(b_j!) \geq s \ln\left(s^{-1}\sum_{j=1}^{s} b_j\right)!$$

This equation clearly holds, by Jensen's inequality, once we show that $\ln x!$ is a convex function of x. The second derivative of $\ln\Gamma(x)$ is given by the *trigamma function*, with well-known representation:

$$\frac{d^2}{dx^2}\ln\Gamma(x) = \sum_{j=0}^{\infty}\frac{1}{(j+x)^2},$$

a quantity which is clearly positive for $x > 0$. Therefore, we may conclude that (17) holds.

To simplify our calculations, we note that $r_0 = 1$ and that we can extend our rank sequence by defining $r_{n(t-1)+1}, \ldots, r_{tn} = 0$, allowing us to apply our bound for $r_1, \ldots, r_{tn}$, yielding

$$L([t]^n) \geq \prod_{j=1}^{n(t-1)} r_j! \geq \left(\left(\frac{1}{tn}\sum_{j=1}^{nt}|[t]^n|\right)!\right)^{tn} \geq \left(\frac{t^{n-1}}{ne}\right)^{t^n},$$

with the final calculation following from Stirling's formula. This establishes the bound in Proposition 3.5.

In order to establish an upper bound on $L([t]^n)$, we will need a good approximation of the size of the levels of $[t]^n$. The (ordinary) generating function of the rank sequence of $[t]^n$ is given by

$$f(x) = \sum_{s=0}^{n(t-1)} r_s x^s = \left(\sum_{s=0}^{t-1} x^s \right)^n .$$

Hence, we can think of f as t^n times the probability generating function for the distribution arising from the n-fold convolution of a uniform distribution on $\{0, \ldots, t-1\}$. As long as n tends to infinity suitably quickly compared to t, this convolution is approximately normal, with mean $n(t-1)/2$ and variance $n(t^2-1)/12$. Indeed, in [3], the authors show that, assuming as before that $t = o(n^{1/8})$,

$$r_s \leq t^n \sqrt{\frac{6}{n(t^2-1)\pi}} \exp\left(\frac{-3(2s - n(t-1))^2}{2n(t^2-1)} \right) + O\left(\frac{t^n}{n\sqrt{t^2-1}} \right). \quad (18)$$

We will use c_1 to represent the amplitude of this normal distribution, i.e.

$$c_1 = t^n \sqrt{\frac{6}{n(t^2-1)\pi}}.$$

Additionally, we will abbreviate the argument of the exponential term as:

$$d_s = \frac{-3(2s - n(t-1))^2}{2n(t^2-1)}.$$

Proposition 3.6 *For $t = o(n^{1/8})$,*

$$L([t]^n) \leq \exp\left(t^n \left((n-1)\ln t + O(n^{1/4}\sqrt{\ln t}) \right) \right),$$

To establish this bound, we will appeal to Theorem 3.3. Our first step is to use (18) to provide an estimate for $r_s \ln r_s$. We will use the observation that $\ln(x+y) \leq \ln x + y/x$. We therefore estimate that:

$$\begin{aligned} r_s \ln r_s \leq & \left(c_1 \exp(d_s) + O\left(\frac{t^n}{n\sqrt{t^2-1}} \right) \right) \\ & \cdot \left(n \ln t + \frac{1}{2} \ln\left(\frac{6}{n(t^2-1)\pi} \right) - d_s + O\left(\sqrt{\frac{6}{n\pi}} \right) \exp(d_s) \right). \end{aligned}$$

To compute $\sum_{s=0}^{n(t-1)} r_s \ln r_s$, we will consider the behavior of the sum in 3 different ranges. The bulk of the weight of the sum occurs around the center level of the poset, when $s = \frac{n(t-1)}{2}$. We will take a band of levels above and below the center with radius R given by:

$$R = \sqrt{\frac{\ln n}{8} + \frac{\ln \ln t}{12}} \sqrt{n(t^2-1)}.$$

We can bound $\ln r_s$ in the range $\left(\frac{n(t-1)}{2} - R, \frac{n(t-1)}{2} + R\right)$ by the term:

$$\begin{aligned} c_2 &= n \ln t + \frac{1}{2} \ln\left(\frac{6}{n(t^2-1)\pi}\right) + O\left(\sqrt{\frac{6}{n\pi}}\right) \exp\left(\frac{3(2R)^2}{2n(t^2-1)}\right) \\ &= (n-1)\ln t + O(n^{1/4}\sqrt{\ln t}). \end{aligned}$$

From here, we can bound each of the partial sums in turn:

$$\begin{aligned} \sum_{s=0}^{n(t-1)} r_s \ln r_s &\leq \sum_{s=0}^{n(t-1)/2-R} r_s \ln r_s + c_1 c_2 \Delta_1 + O\left(\frac{t^n}{n\sqrt{t^2-1}}\right) c_2 + \sum_{s=n(t-1)/2+R}^{n(t-1)} r_s \ln r_s \\ &= c_1 c_2 \Delta_1 + 2 \sum_{s=0}^{n(t-1)/2-R} r_s \ln r_s + O\left(\frac{t^n \ln t}{\sqrt{t^2-1}}\right), \end{aligned}$$

Where $\Delta_1 = \sum_{s=0}^{n(t-1)} \exp(d_s)$.

First of all, we note the the tails of the sum are symmetric and we can use a standard Chernoff-type bound to measure their contribution.

$$\begin{aligned} \sum_{s=0}^{n(t-1)/2-R} r_s \ln r_s &\leq n \ln t \sum_{s=0}^{n(t-1)/2-R} r_s \\ &\leq 2 \ln t \cdot t^n \cdot n \exp\left(\frac{-6R^2}{n(t^2-1)}\right) \\ &= 2 \ln t \cdot t^n \cdot n \ \exp\left(-3\ln n/4 - \ln\ln t/2\right) \\ &= O(t^n n^{1/4}\sqrt{\ln t}). \end{aligned}$$

To complete our proof, we just need to provide a bound for Δ_1. We can think of this term as a Riemann sum approximating the integral of an appropriately chosen function. Let $u = (2s - n(t-1)) \cdot \sqrt{\frac{3}{2n(t^2-1)}}$; then we see that

$$\begin{aligned} \Delta_1 &= \sum_{s=0}^{n(t-1)} \exp\left(\frac{-3(2s-n(t-1))^2}{2n(t^2-1)}\right) && (19) \\ &= \sqrt{\frac{n(t^2-1)}{6}} \left(\int_{-\infty}^{\infty} e^{-u^2}\, du + O(n^{-2}t^{-2})\right) && (20) \\ &= \sqrt{\frac{n(t^2-1)}{6}} \cdot \sqrt{\pi} + O(n^{-3/2}t^{-1}) && (21) \\ &= \sqrt{\frac{\pi n(t^2-1)}{6}} + O(n^{-3/2}t^{-1}), && (22) \end{aligned}$$

In (20), note both the error term resulting from the conversion between Riemann sum and integral and the normalization factor. Combining all of these bounds, and expanding the product $c_1c_2\Delta_1$ we establish the result of Proposition 3.6:

$$\sum_{s=0}^{n(t-1)} r_s \ln r_s \leq t^n \left((n-1)\ln t + O(n^{1/4}\sqrt{\ln t}) \right).$$

Comparing the first order term with the lower bound above gives us the result of Theorem 1.2. Indeed, we have

$$\prod_{s=0}^{n(t-1)} r_s^{r_s} \leq \exp\left(t^n \left((n-1)\ln t + O(n^{1/4}\sqrt{\ln t}) \right) \right),$$

which provides an upper bound on the number of linear extensions of $[t]^n$, by Theorem 3.3, while the lower bound is of the form

$$\begin{aligned} \left(\frac{t}{(ne)^{1/(n-1)}} \right)^{(n-1)t^n} &= \exp\left(t^n (n-1)\ln t - t^n \ln(ne) \right) \\ &= \exp\left(t^n \left((n-1)\ln t - \ln n - 1 \right) \right). \end{aligned}$$

3.2 Linear Extensions of $F_{n,k}$

Recall that for $F_{n,k}$, the number of levels in the poset is $K = n+1$, the rank function is $r_i = \binom{n}{i}k^i$, and $|F_{n,k}| = (k+1)^n$. For ease of notation let

$$\mathcal{L} = \frac{1}{(k+1)^n} \log\left(\prod_{i=0}^{n} \left(\binom{n}{i} k^i \right)! \right).$$

Theorem 3.7

$$\mathcal{L} \leq \frac{\log L(F_{n,k})}{|F_{n,k}|} \leq \mathcal{L} + O\left(\frac{\log n}{n} \right).$$

More specifically,

$$n\log(k+1) - \log(n+1) - \log e \leq \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \leq n\log(k+1) - \log e + o(1).$$

Proof. Using the results from (14) and Theorem 3.4, for a ranked poset P with levels indexed by $1, \ldots, K$ and rank sequence $\{r_i\}_{i=1}^K$ we know that:

$$\frac{1}{|P|}\log\prod_{i=1}^{K} r_i! \leq \frac{1}{|P|}\log L(P) \leq \frac{1}{|P|}\log\left(\prod_{i=1}^{K} r_i! \, \frac{2e(K-1)N}{r} \right)^r. \qquad (23)$$

The bound in the second half of Theorem 3.7 follows by giving an expansion for $\mathcal{L}$ since this term appears in both the upper and lower bounds.

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{(k+1)^n} \log \prod_{i=-0}^{n} \left(\binom{n}{i} k^i \right)! \\
&= \sum_{i=0}^{n} \frac{\log \left(\sqrt{2\pi \binom{n}{i} k^i} \left(\frac{\binom{n}{i} k^i}{e} \right)^{\binom{n}{i} k^i} (1 + o(1)) \right)}{(k+1)^n} \\
&= \sum_{i=0}^{n} \frac{\binom{n}{i} k^i \log \left(\binom{n}{i} k^i \right)}{(k+1)^n} - \log e + o(1).
\end{aligned} \tag{24}
$$

To bound the sum in this term, we use standard techniques: Stirling's formula, the derivative of the Binomial Theorem, bounds for binomial coefficients, and Jensen's inequality. We see that

$$
-\log \left(\frac{1}{(k+1)^n} (n+1) \right) \leq \sum_{i=0}^{n} \frac{\binom{n}{i} k^i \log \left(\binom{n}{i} k^i \right)}{(k+1)^n} \leq \left[\frac{k \log k}{k+1} + h \left(\frac{k}{k+1} \right) \right] n \tag{25}
$$

Here the function $h(x)$ is the binary entropy function.

The result follows once it is established that the error term in the upper bound from the additional term in the Brightwell-Tetali bound (16), namely

$$
\frac{\log \left(\left(\frac{2e(K-1)N}{r} \right)^r \right)}{|F_{n,k}|} = \frac{r \log \left(\frac{(k+1)^n}{r} \right) + r \log(2en)}{(k+1)^n}, \tag{26}
$$

has a smaller order than the linear order of the leading terms; in particular we can see that it has order $O\left(\frac{\log n}{n}\right)$. First we find the value for the degree parameter r described above.

$$
r = \sum_{i=1}^{n} \frac{r_{i-1}}{d_i} = \sum_{j=0}^{n-1} \frac{k^j \binom{n}{j}}{j+1} = \frac{(k+1)^{n+1} - 1}{k(n+1)} - \frac{k^n}{n+1} = \frac{(k+1)^{n+1} - k^{n+1} - 1}{k(n+1)}. \tag{27}
$$

In the simplification of this term, we use a standard equation with closed form given in Page 34 of [13].

Using the value for r from (27) in expression (26), and combining the results from (24), and (25), we establish the bounds:

$$
n \log(k+1) - \log(n+1) - \log e \leq \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \leq n \log(k+1) - \log e + o(1),
$$

as desired.

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