The Slope Polynomial and Collinear Points in Permutations

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Abstract
We consider a polynomial – which we term the “slope polynomial” – that encodes information about slopes of lines defined by a point-set in finite affine planes. When the aforementioned point-set is the graph of a permutation, we show that constancy of the polynomial is equivalent to the permutation being linear. This has immediate consequences for the structure of the 3-uniform hypergraph of collinear triples of the graph of a permutation, an issue that is itself connected with important questions in harmonic analysis and combinatorial number theory.

1 Introduction
Let σ be a permutation of $\mathbb{F}_q$. In this note, we examine the “slope polynomial” $\phi_\sigma : \mathbb{F}_q^2 \to \mathbb{F}_q$, that is, the $u$-linear coefficient of what S. Ball terms the “direction polynomial” (q.v. [1], up to a slight change in notation)

$$F(u, t) = \prod_{j \in \mathbb{F}_q} (1 - (j + \sigma(j)x - t)^{q-1}u),$$

where $x \in \mathbb{F}_q[x]/\langle x^2 - r \rangle$ for some quadratic non-residue $r \in \mathbb{F}_q$. (We ignore the case when $q$ is a power of two, since, for applications, there turns out to be a different story entirely.) The function $F$ has played a crucial role in several aspects of finite geometry, beginning with Rédei’s milestone...
paper [4] on lacunary polynomials and continuing today in the work of Ball, Blokhuis, Gács, Mazzocca, Storme, Szőnyi, and many others on intersection sets, blocking sets, and related questions.

We show, in particular, that, if $\phi_\sigma$ is constant on a set of size at least $q - 2$, then $\sigma$ is linear. This has an immediate application to structural questions about the hypergraph of collinear triples in the graph of a permutation (see [2] or below for definitions) and hence indirect application to the finite field Kakeya problem (q.v. [3]), a question of great import in harmonic analysis and additive number theory (q.v. [5]).

Let $q$ be an odd prime power. Write $\mathbb{F}_q$ for the finite field of order $q$ and $\mathbb{F}_q^*$ for its multiplicative group, i.e., $\mathbb{F}_q \setminus \{0\}$. Given $P \subset \mathbb{F}_q^2 = \mathbb{F}_q \times \mathbb{F}_q$, define the 3-uniform hypergraph $\mathcal{C}(P)$ on the vertex set $P$ to be the set of all $\mathbb{F}_q$-collinear triples $\{p_1, p_2, p_3\} \subset P$. Let $f : \mathbb{F}_q \to \mathbb{F}_q$ be some function; then we define the “graph of $f$”, denoted $\Gamma_f$, to be the set $\{(j, f(j)) : j \in \mathbb{F}_q\} \subset \mathbb{F}_q^2$. We will write $m_{ij} = m_{ij}(f) \in PG(q, 1)$ for the slope of the line connecting the points in $\Gamma_f$ corresponding to the ordinates $i$ and $j$, i.e., $m_{ij} = (f(j) - f(i))/(j - i)$. We refer to the slope of a collinear triple in the obvious sense. In general, a slope can take on the value $\infty$ if the denominator of this expression is 0, although this is clearly never the case for a graph $\Gamma_f$. Also, the slope of a triple cannot be 0 for $\sigma$ a permutation. Finally, we write $\ell(x, m)$ for the line in $\mathbb{F}_q^2$ through $x$ with slope $m$.

The next section contains our main theorem and its consequences; the following section discusses the related issue of the group structure on $PG(q, 1)$ induced by a linear embedding $\mathbb{F}_q^2 \hookrightarrow \mathbb{F}_q^2$. We conclude this section with an intriguing open question.

**Conjecture 1.** For any permutation $\sigma : \mathbb{F}_q \to \mathbb{F}_q$, $|\mathcal{C}(\Gamma_\sigma)| \geq (q - 1)/2$.

Note that, if this conjecture is true, it is best possible, as evidenced by the graph of the function $s \mapsto s^{-1}$ for $s \neq 0$ and $0 \mapsto 0$.

## 2 Constancy of the Slope Polynomial

Suppose that we embed $\mathbb{F}_q^2$ into $\mathbb{F}_q^2$ via any vector-space isomorphism $\psi$. Given any two $p_1, p_2 \in \mathbb{F}_q^2$, the quantity

$$\psi(p_1 - p_2)^{q-1}$$
is a \((q + 1)\)-st root of unity in \(\mathbb{F}_{q^2}\). Furthermore, it is easy to see that, if \(p_3 \in \mathbb{F}_{q^2}^2\),

\[
\psi(p_1 - p_2)^{q-1} = \psi(p_2 - p_3)^{q-1}
\]

iff \(p_1, p_2,\) and \(p_3\) are collinear. Since each of the \(q + 1\) such roots of unity arise in this manner, they correspond exactly to the \(q + 1\) slopes \(PG(1,q) = \mathbb{F}_q \cup \{\infty\}\). Denote by \(\omega_m\) the \((q + 1)\)-st root corresponding to the slope \(m\).

It will be convenient to fix a mapping \(\psi\). Hence, we consider \(\mathbb{F}_{q^2} \cong \mathbb{F}_q[x]/(x^2 - r)\), where \(r\) is a quadratic non-residue of \(\mathbb{F}_q\), and let \(\psi(1,0) = 1\) and \(\psi(0,1) = x\). Note that this implies that

\[
\omega_0 = \psi(1,0)^{q-1} = 1^{q-1} = 1,
\]

and

\[
\omega_\infty = \psi(0,1)^{q-1} = r^{q-1} = -1.
\]

Finally, let \(a_j = \psi(j,\sigma(j))\), and consider the polynomial \(\phi_\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}\), which we refer to as the “slope polynomial,” defined by

\[
\phi_\sigma(t) = \sum_{i \in \mathbb{F}_q} (t - a_i)^{q-1}.
\]

(1)

**Lemma 1.** \(\sum_{m \in PG(1,q)} \omega_m = 0\).

**Proof.** \(\sum_{m \in PG(1,q)} \omega_m = \sum_{m \in PG(1,q)} (\omega_\infty \omega_m) = -\sum_{m \in PG(1,q)} \omega_m.\) \(\square\)

**Theorem 2.** \(\phi_\sigma\) is constant iff \(\sigma\) is linear.

**Proof.** First, note that it suffices to consider the case when \(\sigma(0) = 0\). To see sufficiency, assume that \(\sigma\) is linear of slope \(m\). Observe that, when \(t \in \Gamma_\sigma\), \(\phi_\sigma(t) = (q-1)\omega_m = -\omega_m\). On the other hand, when \(t \not\in \Gamma_\sigma\), there is exactly one line through \(t\) which does not intersect \(\Gamma_\sigma\) (namely, \(\ell(t,m)\)), and every other slope \(m'\) has the property that \(|\ell(t,m') \cap \Gamma_\sigma| = 1\). Hence, the sum \(\phi_\sigma(t)\) contains exactly one summand \(\omega_{m'}\) for each such \(m'\), and, by Lemma 1, \(\phi_\sigma(t) = -\omega_m\). Hence, \(\phi_\sigma(t)\) does not depend on \(t\).

For necessity, let \(b_j \in \mathbb{F}_{q^2}, 0 \leq j \leq q - 1\), be defined by

\[
\phi_\sigma(t) = \sum_{j=0}^{q-1} b_j t^j.
\]
If $\phi^\sigma$ is constant, then $b_j = 0$ for $1 \leq j \leq q - 1$. By the definition of $\phi^\sigma$,

$$b_j = (-1)^j \binom{q-1}{j} \sum_{i \in \mathbb{F}_q} a_i^j.$$ 

It is easy to see that $(q^{-1}) \equiv (-1)^j \pmod{p}$, where $p$ is the characteristic of $\mathbb{F}_q$, so we may conclude that

$$\sum_{i \in \mathbb{F}_q^*} a_i^j = 0$$

when $1 \leq j \leq q - 1$.

Now, we may write $a_i = i + x^j(i)$ for each $i \in \mathbb{F}_q$, where $x$ is such that $\mathbb{F}_{q^2} \cong \mathbb{F}_q[x]/(x^2 - r)$. Then, for each $j \in [q - 1]$,

$$\sum_{i \in \mathbb{F}_q^*} a_i^j = \sum_{i \in \mathbb{F}_q^*} (i + x^j(i))^j$$

$$= \sum_{i \in \mathbb{F}_q^*} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} r^k \sigma(i)^k$$

$$= \sum_{i \in \mathbb{F}_q^*} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} r^{\lfloor k/2 \rfloor} x^k \mod 2 \sigma(i)^k$$

$$= \sum_{k=0}^{\lfloor j/2 \rfloor} \sum_{i \in \mathbb{F}_q^*} \binom{j}{2k} i^{j-2k} r^k \sigma(i)^{2k}$$

$$+ x \sum_{k=1}^{\lfloor j/2 \rfloor} \sum_{i \in \mathbb{F}_q^*} \binom{j}{2k-1} i^{j-2k+1} r^{k-1} \sigma(i)^{2k-1}.$$ 

Since $\mathbb{F}_{q^2}$ is a vector space over $\mathbb{F}_q$ spanned by 1 and $x$, the fact that this expression is zero implies that each of the two outer summands is zero, i.e.,

$$\sum_{k=1}^{\lfloor j/2 \rfloor} \sum_{i \in \mathbb{F}_q^*} \binom{j}{2k-1} i^{j-2k+1} r^{k-1} \sigma(i)^{2k-1} = 0.$$ 

Rewriting this and defining $J = \lfloor j/2 \rfloor - 1$, we have

$$j \sum_{i \in \mathbb{F}_q^*} i^{-1} \sigma(i) + r^j \sum_{i \in \mathbb{F}_q^*} i^{-3} \sigma(i)^3 +$$
\[ \ldots + r^J \left( \frac{j}{2J+1} \right) \sum_{i \in \mathbb{F}_q^*} i^{j-2J-1} \sigma(i)^{2J+1} = 0. \]  

(2)

Since we are free to choose \( r \) to be any quadratic non-residue of \( \mathbb{F}_q \), we can isolate the terms of this expression. Indeed, the Vandermonde matrix

\[
\text{VDM}(r_1, \ldots, r_{J+1}) = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
 r_1 & r_2 & \cdots & r_J & r_{J+1} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 r_1^{J-1} & r_2^{J-1} & \cdots & r_J^{J-1} & r_{J+1}^{J-1} \\
 r_1^J & r_2^J & \cdots & r_J^J & r_{J+1}^J
\end{bmatrix}
\]

is nonsingular exactly when \( r_1, \ldots, r_{J+1} \) are distinct. It is possible to choose \( \lceil j/2 \rceil \) distinct quadratic non-residues from \( \mathbb{F}_q \) so long as \( j \leq q-1 \), and so we may conclude that each summand in (2) is zero. In particular,

\[ \sum_{i \in \mathbb{F}_q^*} i^j \sigma(i) = 0 \]  

(3)

for each \( j = 0, \ldots, q-2 \). Consider the matrix \( M \) consisting of the first \( q-1 \) rows of the nonsingular matrix \( \text{VDM}(1, \ldots, q-1) \). \( M \) has rank \( q-1 \), and so any linear family of solutions to the system given by (3) (whose variables are the \( \sigma(i) \) for \( i \in \mathbb{F}_q^* \)) is unique in being a solution. We claim that \( \sigma(i) = \lambda \cdot i \) for \( \lambda \in \{0, \ldots, q-1\} \) forms such a linear family. Indeed,

\[ \sum_{i \in \mathbb{F}_q^*} i^j (\lambda \cdot i) = \lambda \sum_{i \in \mathbb{F}_q^*} i^{j+1} = 0 \]

for any \( j \) with \( q \nmid j \), since then the sequence \((i^{j+1})_{i \in \mathbb{F}_q^*}\) is just a permutation of \( \mathbb{F}_q^* \).

\[ \square \]

**Proposition 3.** For \( j \in \mathbb{F}_q \), define

\[ S_j = \left\{ m_{ij} = \frac{\sigma(j) - \sigma(i)}{j - i} : i \neq j \right\}, \]

the multiset of slopes through \((j, \sigma(j))\) and the other points of \( \Gamma_\sigma \). If \( q-2 \) of the \( S_j \) are the same, then \( \sigma \) is linear.

5
Proof. For any \( j \), \( \phi_\sigma(a_j) = \sum_{m \in S_j} \omega_m \). If there are \( R \) points \( a_j \) with the same \( S_j \), then the polynomial \( \phi_\sigma(t) \) has at least \( R \) points where it takes on the same value. We now argue that \( \deg(\phi_\sigma) \leq q - 3 \), so that \( R \leq q - 3 \).

Defining \( b_j \in \mathbb{F}_{q^2} \) as above to be the coefficient of \( t^j \) in \( \phi_\sigma \), and appealing to (1), we see that

\[
\begin{align*}
b_{q-1} &= q = 0, \\
b_{q-2} &= -\sum_{i \in \mathbb{F}_q} a_i = -\sum_{i \in \mathbb{F}_q} (i + x\sigma(i)) = -\sum_{i \in \mathbb{F}_q} i - x \sum_{i \in \mathbb{F}_q} \sigma(i) = 0.
\end{align*}
\]

Hence, \( \deg(\phi) < q - 2 \). The only remaining possibility is that \( \phi_\sigma \) is constant, which by Theorem 2 implies that \( \sigma \) is linear.

3 The Slope Group

The multiplication group of \((q + 1)^{st}\) roots of unity is just \( \mathbb{Z}_{q+1} \). Therefore, defining \( m_1 \ast m_2 \) by \( \omega_{m_1} \ast \omega_{m_2} = \omega_{m_1 \ast m_2} \) gives rise to a (non-canonical) isomorphism between \( PG(1,q) \) (endowed with \( \ast \)) and \( \mathbb{Z}_{q+1} \). The next result describes exactly how \( \ast \) acts on \( PG(1,q) \).

**Proposition 4.** For \( m_1, m_2 \in \mathbb{F}_q \),

\[
m_1 \ast m_2 = \begin{cases} 
\frac{m_1 + m_2}{1 + m_1 m_2 r} & \text{if } m_1 m_2 \neq -r^{-1} \\
\infty & \text{otherwise}
\end{cases}
\]

Furthermore, \( \infty \ast m = m \ast \infty = (rm)^{-1} \) for \( m \in \mathbb{F}_q^* \), \( \infty \ast 0 = 0 \ast \infty = \infty \), and \( \infty \ast \infty = 0 \).

**Proof.** Suppose that \( m_1 m_2 \neq -r^{-1} \). Then

\[
\omega_{m_1} \cdot \omega_{m_2} = (1 + m_1 x)^{q-1} (1 + m_2 x)^{q-1} = ((1 + m_1 m_2 r) + (m_1 + m_2)x)^{q-1}.
\]

Since \( \lambda^{q-1} = 1 \) whenever \( \lambda \in \mathbb{F}_q^* \), we may multiply the right-hand side by \((1 + m_1 m_2 r)^{-(q-1)}\) and get that

\[
\omega_{m_1} \cdot \omega_{m_2} = \left(1 + \frac{(m_1 + m_2)x}{1 + m_1 m_2 r}\right)^{q-1} = \omega_{(m_1 + m_2)/(1 + m_1 m_2 r)}.
\]
On the other hand, if $m_1m_2 = -r^{-1}$, then
\[
\omega_{m_1} \cdot \omega_{m_2} = (1 + m_1x)^{q-1}(1 + m_2x)^{q-1} \\
= ((m_1 + m_2)x)^{q-1} \\
= (m_1 + m_2)^{q-1}r^{(q-1)/2} = 1 \cdot (-1) = -1 = \omega_{\infty}.
\]
To see that $\infty \ast m = m \ast \infty = (rm)^{-1}$, observe
\[
\omega_{m} \omega_{\infty} = \omega_{m}(\omega_{-m}(\omega_{rm})^{-1}) = \omega_{m \ast (-m)}\omega_{(rm)^{-1}} = \omega_{(rm)^{-1}}.
\]
Finally, $\omega_{\infty} \omega_0 = -1 \cdot 1 = \omega_{\infty}$, whence $\infty \ast 0 = 0 \ast \infty = \infty$, and
\[
\omega_{\infty} \omega_{\infty} = -1 \cdot -1 = 1 = \omega_0,
\]
whence $\infty \ast \infty = 0$.

\textbf{Corollary 5.} $\omega_m = (\omega_{-m})^{-1}$.

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\section*{References}


