Sequences and Series

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Overview

Sequences and series are the objects of interest for the next few weeks. The intent of this lab is to provide additional practice determining the convergence or divergence of a sequence of numbers. Ways to generate sequences and series in Maple are also introduced.

Maple Essentials

- New Maple commands introduced in this lab:

Command/Example	Description

`seq(f(n), n=i..j);`
Examples: `seq(1/n,n=1..10);`
`a:=seq(k!,k=3..10); a[3]; a[8];`
`seq(b[m]=sin(m*Pi/2),m=0..8);`
`pt:=seq([n,f(n)],n=1..8); pt[2];`
	It creates a finite sequence of values $f(i)$, $f(i+1)$, $\cdots$, and $f(j)$, where $f(n)$ is a maple function and $i \le j$ are integers. A sequence of points on the graph of $y = f(x)$ can be obtained using:
	> `seq([n,f(n)], n=i..j);`

`sum(f(n), n=i..j)`
Examples: `sum(n^2, n=-1..10);`
`sum((1/2)^n,n=0..infinity);`
`sum(c/k^2,k=1..infinity);`
`S:=n->sum(k,k=1..n); S(n); S(8);`
`f:=x->sum(x^n/n!,n=0..infinity);`
`f(x); f(1); f(-1);`
	It creates and evaluates a finite or infinite sum, that is, series $\sum_{n=i}^{j} f(n)$, where `f(n)` is a maple function or expression and $i \le j$ can be integers, variables, or infinity. For a finite or convergent infinite series, it automatically evaluates the sum and returns a value or formula. If you don't want the automatic evaluation, use `Sum` instead of `sum`.

`for n from i to j do...end do;`
Examples: `for n from 5 to 10 do c[n]:=1/n end do;`
`for n from 0 to 9 do d[2*n]:=1; d[2*n+1]:=0 end do;`
`s[1]:=1; for n from 2 to 8 do s[n]:= s[n-1]+n end do;`
	A typical for-loop (`for` and `do` statement) used in general programming languages. It executes whatever between ``do'' and ``end do'' repeatedly for a counted number of times (``for n from i to j''). It hence can be used to work with sequences in much more general ways than what the command `seq` could.

- A link to the *SequenceDrill* maplet can be found on the course website:

`http://www.math.sc.edu/calclab/142L-S07/labs/`

Preparation

§10.1, §10.2, and §10.3. In addition, review the basic qualitative properties of logarithms, powers, exponentials, and so on. For example, exponentials grow faster (at ∞) than polynomials, factorials grow faster than exponentials, and so on.

Assignment

Exercises 21, 39 on pages 634-635, and 29 on page 642. Please refer to and do activities on the back of this page first.

Activities

1. For each of the following sequences, generate the first 10 terms and determine if it diverges or converges to a limit. (Let p be a parameter.)

   (1) $\{1+(-1)^n\}_{n=1}^{\infty}$ (2) $\{(-1)^n \arctan(n)\}_{n=1}^{\infty}$ (3) $\left\{\sqrt{n^2+pn}-n\right\}_{n=1}^{\infty}$

   (4) $\left\{\frac{10^n}{n!}\right\}_{n=0}^{\infty}$ (5) $\left\{n\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (6) $\left\{\ln(\frac{1}{n})\right\}_{n=1}^{\infty}$

   (7) $\left\{\frac{3+n^2\sin(n)}{2+n^2}\right\}_{n=1}^{\infty}$ (8) $\left\{\sum_{k=1}^{n}\frac{1}{k^2}\right\}_{n=1}^{\infty}$ (9) $\left\{\sum_{k=1}^{n}\frac{1}{1+(k/n)}\right\}_{k=1}^{\infty}$

   Note: *You may use the* SequenceDrill *maplet. However, it does not work well with sequences involving parameters since it involves* `plot`*. We will work out some examples using explicit commands. Notice that there is no direct way to define a sequence in general formulas in Maple and we are really finding limits of functions. There are cases that* $\lim_{n\to\infty} f(n)$ *exists as a sequence but does not exist as a function (eg.* $f(n)=\sin(n\pi)$*). One way to deal with this is to define sequences using* `f(floor(n))` *instead of* `f(n)`*, where* `floor(n)` *is the maple function that returns the largest integer* $k \le n$*.*

2. A typical format for a recursively-defined sequence is $a_{n+1}=f(a_n)$, $n=2,3,\cdots$ (with a_1 given explicitly). Under the assumptions that (i) $\{a_n\}$ converges to L and (ii) f is continuous function (at L), we have that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ and
$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$
   Thus, L must be a solution to $L=f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (numerically, graphically, or exactly).

   (a) (See Exercise 38 on page 634) Consider the sequence $\{a_n\}$ defined by $a_1=\sqrt{6}$, $a_{n+1}=\sqrt{6+a_n}$, $n=1,2,3,\cdots$. Use Maple to verify that it is a bounded monotone sequence and hence converges to a limit. Explain how a plot containing the graphs of $y=x$ and $y=\sqrt{6+x}$ confirms this limit.

   (b) (See Example 10 on page 633) Consider the sequence $\{x_n\}$ produced by Newton's Method to approximate $\sqrt{2}$ as a zero of $f(x)=x^2-2$. From Exercise 21 of §5.6, we have $x_1=1$, $x_{n+1}=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right)$, $n=1,2,3,\cdots$. Use Maple to verify that the limit is indeed $\sqrt{2}$.

Examples: (8) in activity 1 and (a) in activity 2

```
>restart;
>with(plots):
>a8:= n->sum(1/k^2, k=1..n);
>seq(a8(n), n=1..10);
>p8:=evalf(seq([n,a8(n)], n=1..10));
>plot([p8], style=point);
>limit(a8(n), n=infinity);
>a[1]:=sqrt(6);
>for n from 1 to 9 do a[n+1]:=sqrt(6+a[n]); evalf(a[n+1]) end do;
>plot([x,sqrt(6+x)],x=-6..6);
>solve(x=sqrt(6+x),x);
```