Sequences

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Overview

Sequences (and series) are the objects of interest for the next couple of weeks. The intent of this lab is to provide additional practice determining the convergence or divergence of a sequence of numbers. This will be accomplished using a drill maplet and explicit commands in a worksheet.

Maple Essentials

- A link to the *SequenceDrill* maplet can be found on the course website:

  `http://www.math.sc.edu/calclab/142L-F05/`

- New Maple commands introduced in this lab include:

Command	Description
`!`	factorial, e.g., `5!` is 120 (try `100!`)
`->`	the *arrow operator* is used to define a function For example, $a(n) = \sqrt{n^2+n+1} - n$ could be defined with `> a := n -> sqrt(n^2+n+1)-n;` Then, $a(10)$ is `a(10)` and $\lim_{n\to\infty} a_n$ is `limit( a(n), n=infinity );`
`seq`	create a *seq*uence of values If `a` is a Maple function, the values of $a(1)$, $a(2)$, $a(3)$, and $a(4)$ can be obtained, simultaneously, using: `> seq( a(n), n=1..4 );` Points on the graph of a function could be obtained using: `> seq( [n,a(n)], n=1..100 );` **Note:** *Maple's `seq` command is similar to a `for` or `do` statement in other programming languages.*
`sum`	a finite or infinite sum (including infinite series) The sum of the first n positive integers is `> sum( k^2, k=1..n );` and the sum of a convergent geometric series is `> sum( (3/4)^k, k=1..infinity );` Observe that Maple automatically evaluates a `sum`.

Preparation

Review the basic definitions of *convergence* and *divergence* of a sequence. In addition, review the basic qualitative properties of powers, logarithms, and exponentials. For example, any exponential grows faster (at ∞) than any polynomial; factorials grow faster than exponentials.

Assignment

The Activities cover a wide variety of sequences, including sequences with recursively-defined terms. The techniques introduced in the Activities should prove useful in checking your solutions to assigned problems from the text. Mastery Quiz 7 tests these skills.

Activities

1. Use the *SequenceDrill* maplet to determine the convergence or divergence of the following sequences.

   (i) $\left\{\dfrac{n^2+3n-1}{n^2-5n-6}\right\}_{n=7}^{\infty}$ (ii) $\left\{n\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (iii) $\left\{\dfrac{n^3e^n}{\pi^n}\right\}_{n=1}^{\infty}$

   (iv) $\{(-1)^n\arctan(n)\}_{n=1}^{\infty}$ (v) $\left\{\dfrac{10^n}{n!}\right\}_{n=0}^{\infty}$ (vi) $\left\{\dfrac{(-1)^n\sin(n)^2}{n}\right\}_{n=1}^{\infty}$

   Note: *For additional examples, see the Exercises in your textbook or let the maplet generate random sequences.*

2. Determine if the following sequences converge or diverge, and the limit of any convergent sequence. (Let a and b be real numbers.)

   (i) $\left\{\left(\dfrac{n+a}{n+b}\right)^n\right\}_{n=1}^{\infty}$ (ii) $\left\{\sqrt{n^2+an+b}-n\right\}_{n=1}^{\infty}$ (iii) $\left\{an\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$

   (iv) $\left\{\dfrac{1}{k}\right\}_{k=1}^{\infty}$ (v) $\left\{\displaystyle\sum_{k=1}^{n}\frac{1}{1+(k/n)}\right\}_{k=1}^{\infty}$ (vi) $\left\{\displaystyle\sum_{k=1}^{n}\frac{k}{n^2}\right\}_{n=1}^{\infty}$

   (vii) $\left\{\dfrac{3+n^2\sin(n)}{2+n^2}\right\}_{n=1}^{\infty}$ (viii) $\left\{\displaystyle\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(k/n)}\right\}_{n=1}^{\infty}$ (ix) $\left\{\displaystyle\sum_{k=1}^{n}\frac{k^2}{n^3}\right\}_{n=1}^{\infty}$

   Note: *The* SequenceDrill *maplet does not work well with sequences involving parameters. It's probably best to use explicit commands in a worksheet to help with these examples.*

   Example: Activity 2 (vi)

   ```
   >with(plots):
   >a6:= n-> sum(k/n^2, k=1..n );
   >p6:= [seq( [n,a6(n)], n=1..20 )];
   >evalf( p6 );
   >plot( p6, style=point );
   >limit( a6(n), n=infinity );
   >plot( [p6,1/2], n=0..20, style=[point,line], view=0..1 );
   ```

3. A standard format for a recursively-defined sequence is $a_{n+1} = f(a_n)$, $n = 2, 3, \ldots$ (with a_1 given explicitly). Under the assumptions that (i) $\{a_n\}$ converges to L and (ii) f is continuous function (at L), we find that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ and

   $$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

   Thus, L must be a solution to $L = f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (exactly, numerically, or graphically).

   (a) Consider the sequence $\{a_n\}$ defined by $a_1 = 0$, $a_{n+1} = \sqrt{6+a_n}$, $n = 1, 2, 3, \ldots$. Assuming this sequence converges, find the (exact) value of the limit, L. Explain how a plot containing the graphs of $y = x$ and $y = \sqrt{6+x}$ confirms that this equation has a solution.