Taylor Polynomials II

Douglas B. Meade
Department of Mathematics

Overview

This is a continuation of last week's lab. The Activities and Project explore additional applications and manipulations of Taylor polynomials and their remainders. These questions will be helpful as you begin to study power series.

Maple Essentials

There are no new Maple commands to learn this week.

Note: *If you need help with `fsolve`, see Lab C or the online help.*

Preparation

Review last week's lab (Lab I) and the material about Taylor polynomials and remainders.

Assignment

1. Project 2 is due at the beginning of next week's lab. Remember to follow the Project Report Guidelines that were handed out with Lab F (and available on the lab homepage).
2. For Mastery Quiz 8 you will need to answer some questions about Taylor polynomials.

Activities

1. Suppose the 4^{th} Maclaurin polynomial for an unknown function $f(x)$ is
$$p_4(x) = 1 - x + \frac{2}{9}x^2 - \frac{3}{25}x^3 + \frac{4}{49}x^4.$$
Find $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, and $f^{(4)}(0)$.
2. (a) Use the 1^{st} Maclaurin polynomial for $f(x) = \sin(x^2) + \cos(x^2)$ to approximate $\int_{-1}^{1} \sin(x^2) + \cos(x^2)\, dx$. That is, compute $\int_{-1}^{1} p_1(x)\, dx$.
   (b) Use the Remainder Estimation Theorem to find a bound for $R_1(x)$. This result will be of the form: $|R_1(x)| \le \frac{M}{2}|x|^2$ for some value of M.
   (c) Use the result found in (b) to approximate the difference $\int_{-1}^{1} f(x)\, dx - \int_{-1}^{1} p_1(x)\, dx$.
   (d) Repeat (a), (b), and (c) for the n^{th} Maclaurin polynomials for $n = 2$, 4, 8, and 16.
   (e) Compare the number of decimal digits to the right of the decimal point are known to be correct based on the Remainder Estimation Theorem with the actual number of correct digits. **Note:** *Use Maple's value for the integral as the exact value.*
   (f) Find the lowest degree Maclaurin polynomial that can be used to approximate $\int_{-1}^{1} f(x)\, dx$ to six (6) decimal places to the right of the decimal point.
3. Repeat 2. for the integral $\int_{-\pi}^{\pi} e^{-x^2}\, dx$.

Project 2: Optimal Selection of Base Point

A Taylor polynomial is most accurate at its base point ($x = x_0$). If you are interested in using a Taylor polynomial to approximate a function over an interval, the choice of the base point can have a significant role in the overall accuracy of the approximation. In this project you will see how approximations can change for different base points. You will also see how to choose an "optimal" base point.

Let $f(x) = e^{-x^2} + 2\sin(x)$ on the interval $[-\pi, 1]$. Let $p_{n,a}(x)$ denote the n^{th} Taylor polynomial for $f(x)$ about $x = a$, e.g., $p_{2,0}(x)$ is the quadratic Maclaurin polynomial. Lastly, let

$$E_n(a) = \int_{-\pi}^{1} (f(x) - p_{n,a}(x))^2 \, dx.$$

1. Find $p_{2,0}(x)$, $p_{2,-\pi}(x)$, and $p_{2,1}(x)$.
2. Find $a = a^*$, the value of a that minimizes E_2 on $[-\pi, 1]$.

   Hint: *Where can E_2 have a minimum on $[-\pi, 1]$?*
3. Create a single plot showing the graphs of f, $p_{2,0}$, $p_{2,-\pi}$, $p_{2,1}$, and p_{2,a^*} on $[-\pi, 1]$.

   Note: *Choose the vertical window so the 5 curves are clearly visible, even if this means one or more of the graphs extends above or below the window.*
4. Fill in the missing values from the following table.

F	$F(-\pi)$	$F(0)$	$F(1)$	$F(a^*)$	$\int_{-\pi}^{1} F(x)\, dx$
f					
$p_{2,-\pi}$					
$p_{2,0}$					
$p_{2,1}$					
p_{2,a^*}					

5. Repeat 1.–4. using the 3^{rd} Taylor polynomial. That is, replace $p_{2,a}(x)$ with $p_{3,a}(x)$ and $E_2(a)$ with $E_3(a)$.
6. Repeat 1.–4. using the 4^{rd} Taylor polynomial, That is, replace $p_{2,a}(x)$ with $p_{4,a}(x)$ and $E_2(a)$ with $E_4(a)$.

Additional Notes

- You are encouraged to work together to understand the questions in this project. **You must, however, write your own project report.**
- Your report should present the information in items 1–6 above in a logical manner. All of these results should be presented, but not necessarily in the order listed above. Use tables, graphs, and other presentation tools as appropriate and to include explicit references to these from the text.
- The conclusion section of your report should include some comments that compare the results for $n = 2$, $n = 3$, and $n = 4$. For example:
  - Could some of the table entries be filled in *before* you knew a formula for the Taylor polynomials?
  - How do you know you have the *global* minimum of E_n on $[-\pi, 1]$?
  - How do the approximations change as the order of the Taylor polynomial increases?