# Taylor Polynomials

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### *Overview*

A Taylor polynomial for a function is easy to find assuming you (i) understand the concept and related formulae and (ii) can compute the necessary derivatives. It is more difficult to understand the approximation properties of Taylor polynomials. This lab addresses the computation, visualization, and approximation properties of Taylor (and Maclaurin) polynomials.

### *Preparation*

Review the definition of the $n^{\text{th}}$ Taylor (and Maclaurin) polynomial for a function $f$ about $x = x_0$ (with remainder) and the formula for the remainder.

$$
\begin{aligned}
p_n(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n \\
R_n(x) &= f(x) - p_n(x) \\
&= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \qquad \text{for some value of } c \text{ between } x \text{ and } x_0 \\
|R_n(x)| &\le \frac{M}{(n+1)!}|x-x_0|^{n+1} \qquad \text{where } -M \le f^{(n+1)}(c) \le M \text{ for all } c \text{ between } x \text{ and } x_0
\end{aligned}
$$

### *Maple Essentials*

- The *Taylor Approximation* tutor can be started from the Tools menu:

  **Tools → Tutors → Calculus - Single Variable → Taylor Approximation ...**

- New Maple commands introduced in this lab include:

| Command | Description |
|---|---|
| `TaylorApproximation` | numeric or visual representation of one or more Taylor polynomials<br>> `with( Student[Calculus1] ):`<br>> `TaylorApproximation( x*exp(-x), x=0, order=3 );`<br>> `TaylorApproximation( x*exp(-x), x=0, order=3, -1..4, output=plot );`<br>> `TaylorApproximation( x*exp(-x), x=0, order=0..10, -1..4, output=animation );` |
| `add` | computes a finite sum<br>> `add( k^2, k=1..5 )` expands to $1^2+2^2+3^2+4^2+5^2$ (which simplifies to 55)<br>> `add( x^k, k=1..5 )` expands to $x+x^2+x^3+x^4+x^5$ |
| `$` | repetition operator<br>> `x$4` expands to $x, x, x, x$<br>> `x^k $ k=2..4` expands to $x^2, x^3, x^4$ |

## *Assignment*

- Mastery Quiz 7 will be distributed at the end of today's lab. For this quiz, you will need to find a Taylor polynomial and an estimate for the size of the remainder.
- Project 2 will be distributed during next week's lab meeting. The completed project will be due the following week, two weeks from today.

## *Activities*

1. Find, and plot, the Maclaurin polynomials of orders $n = 0, 1, 2, 3, 4, 6, 8$, and 10 for each of the following functions:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (i) | $e^{-x}$ | (ii) | $e^{ax}$ | (iii) | $\cos(\pi x)$ | (iv) | $\sin(\pi x)$ |
| (v) | $\ln(1+x)$ | (vi) | $\dfrac{1}{1+x}$ | (vii) | $x\sin(x)$ | (viii) | $xe^x$ |

2. Find, and plot, the Taylor polynomials of orders $n = 0, 1, 2, 3, 4, 6, 8$, and 10 about $x = x_0$ for each of the following pairs of $f$ and $x_0$:

| | | | | | |
|---|---|---|---|---|---|
| (i) | $f(x) = e^x$, | $x_0 = 1$ | (ii) | $f(x) = e^{-x}$, | $x_0 = \ln 2$ |
| (iii) | $f(x) = \frac{1}{x}$, | $x_0 = -1$ | (iv) | $f(x) = \frac{1}{x+2}$, | $x_0 = 3$ |
| (v) | $f(x) = \sin(\pi x)$, | $x_0 = 1/2$ | (vi) | $f(x) = \cos x$, | $x_0 = \pi/2$ |
| (vii) | $f(x) = \ln x$, | $x_0 = 1$ | (viii) | $f(x) = \ln x$, | $x_0 = e$ |

3. Use the Remainder Estimation Theorem to find an interval containing $x = 0$ over which $f(x)$ can be approximated by $p(x)$ to (at least) three decimal-place accuracy throughout the interval. Check your answer by graphing $|f(x) - p(x)|$ over the interval you obtained.

| | | |
|---|---|---|
| (i) | $f(x) = e^x$, | $p(x) = 1 + x + \dfrac{x^2}{2!}$ |
| (ii) | $f(x) = e^x$, | $p(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$ |
| (iii) | $f(x) = \cos x$, | $p(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$ |
| (iv) | $f(x) = \cos x$, | $p(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$ |
| (v) | $f(x) = \ln(1+x)$, | $p(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ |
| (vi) | $f(x) = \ln(1+x)$, | $p(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$ |
| (viii) | $f(x) = \cos(2x)$, | $p(x) = 1 - 2x^2 + \dfrac{2}{3}x^4 - \dfrac{4}{45}x^6$ |
| (vii) | $f(x) = x\cos x$, | $p(x) = x - \dfrac{x^3}{2} + \dfrac{x^5}{24}$ |

## *Additional Note*

The following explicit Maple commands can be used to find the $n^{\text{th}}$ Taylor polynomial of $f$ at $x = x_0$. (Note that the first term cannot be included inside the sum.)

```
> f := cos(2*x);                                  # define the function to be approximated
> x0 := Pi/4;                                     # base point for the approximation
> n := 20;                                        # order of the approximation
> pn := eval(f,x=x0)                              # Taylor poly: first term (k = 0)
>       + add( eval(diff(f,x$k),x=x0)/k!*(x-x0)^k,
>              k=1..n );                          # Taylor poly: other terms (k > 0)
```