

Power Series

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Overview

There are two fundamental questions to ask about a power series $\sum_{k=0}^{\infty} c_k(x - x_0)^k$:

1. For what values of x does the infinite sum converge?
2. When the series converges, what does it converge to?

The first question is answered with the convergence tests (most often the ratio test for absolute convergence) and special consideration of any endpoints. This convergence can be seen graphically by animating plots of selected partial sums of the series.

The second question is, in general, much more difficult to answer. One of the more useful techniques is to apply common operations (multiplication, division, differentiation, integration, substitution, ...) to a known power series. Maple is much better at this than most of us, but a little practice can quickly improve this skill.

Preparation

Review the definition of interval of convergence for a power series and how the ratio test is used to find the this interval. Also, be able to write down – from memory – the following power series (and their intervals of convergence):

$$\begin{aligned}
 f(x) &= \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k && (|x| < 1) \\
 f(x) &= e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && (|x| < \infty) \\
 f(x) &= \sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} && (|x| < \infty) \\
 f(x) &= \cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} && (|x| < \infty)
 \end{aligned}$$

Maple Essentials

- New Maple commands introduced in this lab include:

Command	Description
<code>display(..., insequence=true)</code>	displays a collection of plots as the frames of an animation; the individual frames can be 2D or 3D plots – without <code>insequence=true</code> all frames are displayed in one plot – the <code>display</code> command is in the <code>plots</code> package

- This lab utilizes the commands introduced in Lab G: the arrow operator (`->`) to create a function, the `sum` and `Sum` commands for defining power series, and the `seq` command for creating a sequence of partial sums and frames in an animation.
- Also, the `solve` command will be used to solve an inequality.

Assignment

- Mastery Quiz 9 will be distributed at the end of today's lab.
- Hour Quiz 3 will be given during next week's lab meeting. This quiz will focus on the material covered since Hour Quiz 2, i.e., Labs I, J, and K.
- Project 3 will be assigned in two weeks (and due in three weeks). This project will involve probability, the error function, and numerical analysis. It will be a nice conclusion to the course.

Activities

1. Consider the power series $\sum_{k=0}^{\infty} x^k$.

- Animate the first 10 partial sums of this series.
- What is the interval of convergence of this series?
- To what function does this series converge? For what values of x ?

2. Repeat Activity 1 for each of the following power series:

(a) $\sum_{k=1}^{\infty} kx^{k-1}$	(b) $\sum_{k=2}^{\infty} k(k-1)x^{k-2}$	(c) $\sum_{k=0}^{\infty} \frac{1}{k+1}x^{k+1}$
(d) $\sum_{k=0}^{\infty} (-x)^k$	(e) $\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$	(f) $\sum_{k=0}^{\infty} \frac{-(-x)^{k+1}}{k+1}$
(g) $\sum_{k=0}^{\infty} \frac{-(-2x)^{k+1}}{k+1}$	(h) $\sum_{k=0}^{\infty} (x-x^2)^k$	(i) $\sum_{k=0}^{\infty} \frac{(x-x^2)^k}{k!}$
(j) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	(k) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}$	(l) $\sum_{k=1}^{\infty} k(k-1)x^{k-1}$

Note: Pay special attention to the power in the first term of each series.

3. The *error function* is defined to be $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ for all x . Note that this integral cannot be expressed in terms of elementary functions. A Maclaurin series expansion for erf can be found by repeated differentiation or by clever use of known Maclaurin series.

- Find the Maclaurin series expansion for e^x . What is the interval of convergence?
- Find the Maclaurin series expansion for e^{-t^2} . What is the interval of convergence?
- Find the Maclaurin series expansion for $\int_0^x e^{-t^2} dt$. What is the interval of convergence?
- Find the Maclaurin series expansion for $\operatorname{erf}(x)$. What is the interval of convergence?
- Let $g(x) = \operatorname{erf}(x)$. Find $g(0)$, $g'(0)$, $g''(0)$, $g^{(5)}(0)$, $g^{(9)}(0)$, and $g^{(10)}(0)$.

Note: Find these values *without* computing any derivatives.

4. Use the ideas in Activity 3 to obtain a Maclaurin series expansions for the following functions:

(a) $F(x) = \int_0^x \frac{dt}{1+t^2}$	(b) $G(x) = \int_0^x \frac{\sin(t)}{t} dt$	(c) $H(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$
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