Definite Integrals and Area

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Overview

Many definite integrals are evaluated using the Fundamental Theorem of Calculus. The process consists of three steps:

STEP 1:	find an antiderivative of the integrand,
STEP 2:	evaluate the antiderivative at the two endpoints,
STEP 3:	subtract the two values found in Step 2.

In Lab B we learned to use the `int` command to evaluate indefinite and definite integrals. Today we will look more closely at definite integrals, particularly Steps 2 and 3.

Maple Essentials

- The *AntiDerivativeDrill* (and *IndefiniteIntegralDrill*) maplets are still good tools to help find antiderivatives. These maplets are available from USC at the URLs:
  `http://www.math.sc.edu/~meade/CalcMaplets/CalcUSC/AntiDerivativeDrill.maplet`
  `http://www.math.sc.edu/~meade/CalcMaplets/CalcUSC/IndefiniteIntegralDrill.maplet`
- New Maple commands introduced in this lab include:

Command	Description
`eval`	`eval(` *F*, *x*`=`*a* `);` evaluates expression F with x replaced by a e.g., `eval( sin(x), x=Pi/2 );` returns 1.
`evalf`	`evalf(` *q* `);` evaluates expression q using floating-point approximations for all numbers and mathematical operations e.g., `evalf( exp( sin(Pi/4) ) );` returns 2.028114981.
`simplify`	`simplify(` *f* `);` simplifies the expression f; e.g., `simplify( exp(a+ln(b*exp(c))) );` returns be^{a+c}.
`=`	*a* `=` *b* is the equation $a = b$ (use `:=` to make an assignment) equations are used in the `solve` command (see below)
`solve`	solve an equation or system of equations: `solve(` *eqn*, *var* `);` solves an equation, *eqn*, for one variable, *var*, `solve( {`*eqn1*,*eqn2*`}, {`*var1*,*var2*`} );` solves a system of two equations for two variables; e.g., `solve( x^4-16=0, x );` returns $2, -2, 2I, -2I$ (complex roots do not appear if the `RealDomain` package is loaded).
`fsolve`	`fsolve(` *eqn*, *var* `);` uses an iterative method, like Newton's Method, to find an approximate solution to the equation; e.g., `fsolve(cos(x)=x/3, x);` returns −2.938100394 and `fsolve(cos(x)=x/3, x=0..Pi);` returns 1.170120950.

Preparation

Review the Fundamental Theorem of Calculus (Part I). A basic understanding of the connection between definite integrals and area between curves will be helpful.

Assignment

For the Mastery Quiz you will need to set up, evaluate, and apply definite integrals and the Fundamental Theorem of Calculus. The deadline for turning in Mastery Quiz 3 will be announced in the lab.

Activities

1. Use the Fundamental Theorem of Calculus to evaluate the following definite integrals in two different ways. First, apply the Fundamental Theorem of Calculus (i.e., show Steps 1–3 above). Second, by direct use of the `int` command.

   (i) $\displaystyle\int_{-1}^{5/3} (x^3 + x - 2)\, dx$ (ii) $\displaystyle\int_{\pi^2/4}^{9\pi^2/16} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$ (iii) $\displaystyle\int_{\sqrt{e}}^{13} t^2 \ln(t)\, dt$

2. Use the following steps to solve Chapter 6 Review Exercise 56 (page 439).
   (a) Define $y = f(x) = x + x^2 - x^3$. **Hint:** *use the assignment* `f := x+x^2-x^3;`
   (b) Find the (exact) locations of the x-intercepts of $f(x)$ **Hint:** *Use the* `solve` *command.*
   (c) Obtain floating-point approximations to the x-intercepts found above.
   Hint: *Use either* `evalf` *or* `fsolve`*.*
   (d) Identify, on a graph, the area in the first quadrant bounded by the graph of $y = f(x)$ and the x-axis.
   (e) Write the area as a definite integral, and evaluate the integral.
   Note: *Is your answer exact or approximate?*

3. Find the area of the region bounded by the graph of $y = x + x^2 - x^3$ and the x-axis.
   Hint: *How is this problem different from Activity 2?*

4. Consider the problem of finding a value for k for which the area above the graph of $y = \sin(x)$ and below the graph of $y = k$ for $0 < x < \frac{\pi}{2}$ equals the area below the graph of $y = \sin(x)$ and above the graph of $y = k$ on the same interval.
   Note: *This is based on §7.1 Exercise 40 on page 449 of the text.*
   (a) Create a graph of $y = \sin(x)$ and $y = \frac{1}{2}$ for $0 < x < \pi$ and identify the regions of interest in this problem.
   (b) With $k = \frac{1}{2}$, formulate the two areas as definite integrals and find the two areas.
   (c) Find the areas of the corresponding regions when $y = \frac{1}{2}$ is replaced with $y = k$.
   Note: *You may assume* $0 < k < 1$*. (Why?)*
   (d) Determine the value of k for which the two areas are equal.

5. Define $F(x) = \displaystyle\int_0^x \frac{1}{t^2+1}\, dt + \int_0^{1/x} \frac{1}{t^2+1}\, dt.$
   Note: *Based on Chapter 6 Review Exercise 66 (page 440).*
   (a) Find exact and approximate values for several positive values of x and several negative values of x. (Why is $F(0)$ not defined?)
   (b) Plot $y = F(x)$ on $-10 < x < 10$.
   (c) Show that $F(x)$ has one value for all $x > 0$, and a different value for all $x < 0$.
   Hint: *Use* `simplify(...) assuming x>0;` *and* `simplify(...) assuming x<0;`

Additional Notes

- Next week's lab (Lab D) will be a one-hour, in-class quiz. The questions on the quiz will test your ability to apply the information, methods, and techniques in the first three Maple labs.