# Project 2: Koch Snowflake and Fractals

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## *Overview*

The word "fractal" is often used in referring any object that is recursively constructed so that it appears similar at all scales of magnification. There are many examples of complex real-life phenomena, such as chaos, ferns, mountains, river networks, biological growth, that can be described and studied using fractals. In this lab and project, we will analyze and generate a classic fractal, the Koch snowflake, and its variations. While it is natural to use a computer to do recursive constructions, we will focus on applications of sequences and series in our study.

## *Koch Snowflake*

1. **Basic Construction:** It starts with an equilateral triangle. A smaller equilateral triangle is then added to each of the three sides. It is done in such a way that the base of each new triangle is the middle one-third of each side of the original figure. This process is repeated again and again in each successive iteration. Notice that while we are getting a sequence of more and more sides, the basic construction is the same: each side is replaced by four sides of one-third of its length. The Koch snowflake is the limit of this construction.
2. **Working with Maple:**
   (a) **Note:** You should always `restart` from the beginning after any modification.
   ```
   > restart;
   > with(plots):
   ```
   (b) **Initial Data:** Each side of the initial equilateral triangle is determined by two of its endpoints (vertices) so we need a sequence of four points with the first and the last being the same vertex.
   ```
   > A[1]:=[0,0];
   > A[2]:=[1/2,sqrt(3)/2];
   > A[3]:=[1,0];
   > A[4]:=A[1];
   ```
   (c) **Initial Results (level 0):** The following should be clear. Notice that `plot` automatically connects a given sequence of points. Ask your TA if you want to know more about the `sprintf` option in `plot`. For example, `8.5f` means floating-point numbers with 8 digits and 5 decimal places.
   ```
   > LenSide[0] := 1;
   > TotArea[0] := LenSide[0]*sqrt(3.)/2.*LenSide[0]/2.;
   > NumSide[0] := 3;
   > TotPerim[0] := NumSide[0]*LenSide[0];
   > plot([A[1],A[2],A[3],A[4]],axes=NONE, scaling=constrained, title=sprintf
   ("Koch snowflake (level 0) \n area = % 8.5f, perimeter = % 8.5f, sides =% d",
   TotArea[0], TotPerim[0], NumSide[0]));
   ```
   (d) **Results up to level n and limits:** Please pay close attention as your TA works out recursive and general formulas for TotArea[n] (a sequence) and TotPerim[n] (a series) by hand. You need to show those details and verify limits in your project report. It is easy to see how many sides are there at level $n$ and it will help you to get other formulas.
   ```
   > n:=10:
   > for k from 1 to n do level:=k;
   >   LenSide[k] := LenSide[k-1]/3.;
   >   TotArea[k] := TotArea[k-1]+NumSide[k-1]*TotArea[0]/9.^k;
   >   NumSide[k] := 4*NumSide[k-1];
   >   TotPerim[k] := NumSide[k]*LenSide[k];
   > end do;
   > TotAreaLimit:=sqrt(3.)*2/5;         # why?
   > TotPerimLimit:=infinity;            # why?
   ```

(e) **Construct sequences of snowflake vertices up to level n:** Start from level 0, between each pair of adjacent vertices $A[m]$ and $A[m+1]$, we need to add three new vertices $B[m]$, $E[m]$, and $C[m]$. (Note: Letter D is reserved in Maple.) We then have to re-index them to form a new sequence of vertices at the next level. Don't worry if you cannot fully follow it now as your TA will help you to complete this part if needed. You need to know how many sides and vertices are there at level n and may want to re-index the vertices explicitly by hand for the first few levels to verify the general formulas.

```
> n:=5:      # Computations for levels above 5 may require too much time and
memory for your computer to handle.  For program testing, use n=2 or 3.
> for k from 1 to n do
>   for m from 1 to 3*4^(k-1) do
>     B[m]:=A[m]+(A[m+1]-A[m])/3;
```

Note that if let A[m+1]-A[m]=[x,y], then [-y,x]=[A[m][2]-A[m+1][2],A[m+1][1]-A[m][1]]. We need this information because if we want to get from the current side out to point E[m], we must travel along a line perpendicular to the side.

```
>     E[m]:=A[m]+(A[m+1]-A[m])/2+sqrt(3)/6.*[A[m][2]-A[m+1][2],A[m+1][1]-A[m][1]];
>     C[m]:=A[m]+2*(A[m+1]-A[m])/3;
>   end do;
>   for m from 1 to 3*4^(k-1) do
>     Temp[4*(m-1)+1]:=simplify(A[m]);
>     Temp[4*(m-1)+2]:=simplify(B[m]);
>     Temp[4*(m-1)+3]:=simplify(E[m]);
>     Temp[4*(m-1)+4]:=simplify(C[m]);
>   end do;
>   for m from 1 to 3*4^k do
>     A[m]:=Temp[m];
>   end do;
>   A[3*4^k+1]:=A[1];
```

(f) **Construct sequence of snowflake plots**

```
>   SFplots[k] := seq(plot([A[m],A[m+1]]),m=1..3*4^k):
> end do:
```

(g) **Display the fractal growth**

```
> for k from 1 to n do
> display([SFplots[k]],axes=NONE, scaling=constrained, title=sprintf
("Koch snowflake (level %a) \n area = %8.5f, perimeter = %8.5f, sides =%d",
k,TotArea[k], TotPerim[k], NumSide[k]));
> end do;
```

**Instructions:** Your report should follow the guidelines set forth in the Project Report Guidelines on the lab website and is due by the date specified by your TA. It should cover the following:

1. Present details to show that you fully understand the given Koch snowflake construction and all the formulas used. In particular, verify in detailed steps the formulas and limits presented here using your knowledge of sequences and series. Have a discussion on what you have learned, such as some interesting properties of the Koch snowflake. For example, as we have computed, the Koch snowflake has a finite area but infinite perimeter. Now, imagining that you have a container with the Koch snowflake as its base and fill it up with some paint, this means you could paint an infinite area (the interior surface of the container) with a finite amount of paint! The Koch snowflake is also an example of a curve that is everywhere continuous but nowhere differentiable.
2. Repeat the same discussion and analysis of at least **two** different variations of the Koch snowflake. Use your imagination but one of them must be constructed as follows: Instead of using triangles, start with a square and recursively add smaller squares. Generate your constructions for the first few levels in Maple. Please ask your TA if you need help. Note: Once you understand the given construction of the Koch snowflake, it should be easily modified for different variations.