

Numerical Integration: Simpson's Rule and Newton-Cotes Formulae

Doug Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

Overview

As we have learned in Calculus I, there are two ways to evaluate a definite integral: using the Fundamental Theorem of calculus or numerical approximations. While FTC is nice in theory, it cannot be applied in many cases, as antiderivatives are often difficult or even impossible to find. Using computers, numerical integration is the most general and practical way to evaluate definite integrals. In Calculus I, we explored Riemann Sum approximations. In this lab, we will use Maple to experience some new approximations that provide more accuracy with less computation.

Maple Essentials

- The *Approximate Integration* tutor can be started from Maple's Tools menu:

Tools → **Tutors** → **Calculus - Single Variable** → **Approximate Integrals ...**

Note: *The Approximate Integration tutor is identical to the Riemann Sums tutor.*

- New Maple commands introduced in this lab include:

Command	Description
<code>ApproximateInt</code>	<p>approximates a definite integral using one of the approximation methods: midpoint, trapezoid or Simpson.</p> <pre>ApproximateInt(f(x), x=a..b, method=method, partition=n);</pre> <p>Note: The <code>ApproximateInt</code> command is in the <code>Student[Calculus1]</code> package.</p>

Related course material/Preparation

Recall that §8.7 of the textbook introduced the trapezoidal approximation as the average of the Riemann left endpoint approximation and the Riemann right endpoint approximation. Similarly, Simpson's rule is introduced there as a weighted average of the Riemann midpoint approximation and the trapezoidal approximation. Geometrically, the trapezoidal approximation and Simpson's rule use a line and a parabola (polynomials of degree 1 and 2), respectively, to approximate $y = f(x)$ in each subinterval. In the trapezoidal approximation, the line is determined by the left and right endpoints of the curve in each subinterval. To determine a parabola, you need three points. Therefore, the left and right endpoints and the midpoint of the curve in each subinterval are used in Simpson's rule. In this point of view, the above two approximations are special cases (order 1 and order 2) of the general Newton-Cotes formula of order k , where a polynomial of degree k determined by $k + 1$ points on $y = f(x)$ is used to approximate the function $y = f(x)$ in each subinterval. Using Maple, we can experience and compare all those approximations with plots, animations, formulas, and more.

Notation

You should be careful about the value n , the number of subintervals. Recall that the notation used in Section 8.7 of the textbook for the Simpson's Rule approximation is S_{2n} . This means that if we want to find S_{10} , we would input $n = 5$ since $2n = 10$.

Activities

Use $n = 10$ subintervals to approximate the integral by (a) the midpoint approximation, (b) the trapezoidal approximation, and use $2n = 10$ subintervals to approximate the integral by (c) Simpson's rule. In each case, use the exact value of the integral (as determined by Maple) to calculate the absolute error. Express your answers to at least four decimal places.

$$1) \int_0^{\pi} \sin x \, dx \quad 2) \int_0^1 e^{-x^2} \, dx \quad 3) \int_1^2 \sqrt{x^3 - 1} \, dx \quad 4) \int_0^2 \sin(x^2) \, dx \quad 5) \int_1^3 \sqrt{\ln x} \, dx$$

Note: These problems appear on pages 566-567 in Anton, Bivens, Davis, 8th Edition.

Example: Activity 1

Using the *Approximate Integration* tutor:

1. Launch the *Approximate Integration* tutor from Maple's Tools menu.
2. Choose **Subintervals** in the **Partition Type** box.
3. Plug in $f(x) = \sin(x)$, $a = 0$, $b = \text{Pi}$, and $n = 10$.
4. Choose your method, say midpoint Riemann sum, and click **Display**. How close is the approximation (the blue graph) to the original curve (the red one)?
5. You should see that the **Approximate Integral** = 2.008248408 and the **Actual Integral** = 2. Therefore, the absolute error is 0.008248408.
6. Change the method to the Trapezoidal rule. Is this method more or less accurate than the midpoint Riemann sum method?
7. Change the method to Simpson's rule. Note that you must change to $n = 5$ since we want $S_{2n} = S_{10}$. Is this method more or less accurate than the previous methods?
8. Choose a value, say $n = 10$, and click on **Compare** to see all approximations at once and compare.
Note: Leave the *normal* box checked in the compare window, so Maple will compare the approximations you have found.

Using Maple commands:

- Start by including the `Student[Calculus1]` package.
`> with(Student[Calculus1]):`
- We define our function and the exact value of the integral so we do not have to type as much later.
`> f:= x -> sin(x);`
`> Actual:= int(f(x),x=0..Pi);`
- For (a) we find the midpoint approximation.
`> Mid:=ApproximateInt(f(x), x=0..Pi, method=midpoint, partition=10);`
- We then find the absolute error associated with this approximation.
`> ErrMid:=abs(Actual - Mid);`
- For (b) we find the trapezoidal approximation.
`> Trap:=ApproximateInt(f(x), x=0..Pi, method=trapezoid, partition=10);`
- We then find the absolute error associated with this approximation.
`> ErrTrap:=abs(Actual - Trap);`
- For (c) we find the Simpson's Rule approximation. Notice that $n = 5$ in this case.
`> Simp:=ApproximateInt(f(x), x=0..Pi, method=simpson, partition=5);`
- We then find the absolute error associated with this approximation.
`> ErrSimp:=abs(Actual - Simp);`

Assignment

Finish lab activities and do exercises 2, 4, and 5 on page 566 of the textbook.