Sequences and Series

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Overview

Sequences and series are the objects of interest for the next few weeks. The intent of this lab is to provide additional practice determining the convergence or divergence of a sequence of numbers. Ways to generate sequences and series in Maple are also introduced.

Maple Essentials

- New Maple commands introduced in this lab:

Command/Example	Description

`evalf(`*expression*`);`
Example:
`evalf(Pi);`
	numerically evaluates expressions involving constants

`seq(f(n), n=i..j);`
Example:
`seq(1/n,n=1..10);`
	creates a finite sequence of values
	$f(i)$, $f(i+1)$, $\cdots$ $f(j)$,
	where $f(n)$ is a maple function and $i \leq j$ are integers.

`seq([n,f(n)],n=i..j);`
Example:
`f:=x->x^2;`
`seq([n,f(n)],n=1..10);`
	creates a finite sequence of points on the graph of $y = f(x)$.

`sum(f(n), n=i..j)`
Example:
`sum(n^2, n=1..10);`
	creates and evaluates a finite or infinite sum, that is, series $\sum_{n=i}^{j} f(n)$, where `f(n)` is a maple function or expression and $i \leq j$ can be integers, variables, or infinity. For a finite or convergent infinite series, it automatically evaluates the sum and returns a value or formula. If you don't want the automatic evaluation, use `Sum` instead of `sum`.

`for n from i to j do...end do;`
Example:
`s[1]:=1; for n from 1 to 9 do`
`s[n+1]:= s[n]+n end do;`
	A typical for-loop (`for` and `do` statement) used in general programming languages. It executes whatever between `''do''` and `''end do''` repeatedly for a counted number of times (`''for n from i to j''`). It hence can be used to work with sequences in much more general ways than what the command `seq` could.

- A link to the *SequenceDrill* maplet can be found on the course website:

`http://www.math.sc.edu/calclab/142L-F07/labs/`→ SequenceDrill

Preparation

§10.1, §10.2, and §10.3. In addition, review the basic qualitative properties of logarithms, powers, exponentials, and so on. For example, exponentials grow faster (at ∞) than polynomials, factorials grow faster than exponentials, and so on.

Assignment

Exercises 21 and 39 on pages 634-635 and exercise 29 on page 642. Please refer to and do activities on the back of this page first.

Activities

1. For each of the following sequences: (a) Generate the first 10 terms. (b) Determine whether the sequence diverges or converges to a limit. (c) Graph a sequence of points to verify your answer. **Note:** *Let p be a parameter.*

   (a) $\{(-1)^n \arctan(n)\}_{n=1}^{\infty}$ (b) $\left\{\sqrt{n^2+pn}-n\right\}_{n=1}^{\infty}$ (c) $\left\{\dfrac{10^n}{n!}\right\}_{n=1}^{\infty}$

   (d) $\left\{n\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (e) $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$ (f) $\left\{\sum_{k=1}^{n}\dfrac{1}{k^2}\right\}_{n=1}^{\infty}$

   (g) $\left\{\sum_{k=1}^{n}\dfrac{1}{1+(k/n)}\right\}_{n=1}^{\infty}$

   Note: *You may use the* SequenceDrill *maplet. However, it does not work well with sequences involving parameters since it involves* `plot`*. We will work out some examples using explicit commands.*

2. A typical format for a recursively-defined sequence is $a_{n+1} = f(a_n)$, $n = 2, 3, \cdots$ (with a_1 given explicitly). Under the assumptions that (i) $\{a_n\}$ converges to L and (ii) f is continuous function (at L), we have that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ and

   $$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

   Thus, L must be a solution to $L = f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (exactly, numerically, or graphically).

   (a) (See Exercise 38 on page 634) Consider the sequence $\{a_n\}$ defined by $a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6+a_n}$, $n = 1, 2, 3, \cdots$. Use Maple to verify that it is a bounded monotone sequence and hence converges to a limit. Explain how a plot containing the graphs of $y = x$ and $y = \sqrt{6+x}$ confirms this limit.

   (b) (See Example 10 on page 633) Consider the sequence $\{x_n\}$ produced by Newton's Method to approximate $\sqrt{2}$ as a zero of $f(x) = x^2 - 2$. From Exercise 21 of §5.6, we have $x_1 = 1$, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$, $n = 1, 2, 3, \cdots$. Use Maple to verify that the limit is indeed $\sqrt{2}$.

Example: Activity 1a

```
> with(plots):
> f:= n-> (-1)^n arctan(n);
> evalf(seq(f(n), n=1..10));
> limit(f(n), n=infinity);
> points:=evalf(seq([n,f(n)], n=1..10));
> P1:=plot([points], style=point):
> P2:=plot([-1/2*Pi, 1/2*Pi]):
> display([P1,P2]);
```

Example: Activity 2a

```
> a[1]:=sqrt(6);
> for n from 1 to 9 do a[n+1]:=sqrt(6+a[n]); evalf(a[n+1]) end do;
> plot([x,sqrt(6+x)],x=-6..6);
> solve(x=sqrt(6+x),x);
```