Taylor Polynomials

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Overview

A Taylor polynomial for a function is easy to find assuming you (i) understand the concept and related formulae and (ii) can compute the necessary derivatives. It is more difficult to understand the approximation properties of Taylor polynomials. This lab addresses the computation, visualization, and approximation properties of Taylor (and Maclaurin) polynomials.

Preparation

Review the following definitions:

1. The nth Taylor polynomial for f about $x = x_0$ (page 679)

$$\begin{aligned} p_n(x) &= f(x_0) + \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \end{aligned}$$

Note: *The special case $x_0 = 0$ is the Maclaurin series for f.*

2. The nth remainder for the Taylor series of f (page 682)

$$\begin{aligned} R_n(x) &= f(x) - p_n(x) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \quad \text{for some value of } c \text{ between } x \text{ and } x_0 \end{aligned}$$

3. The Remainder Estimation Theorem (page 682)
If the function f can be differentiated $n+1$ times on an interval I containing the number x_0, then

$$|R_n(x)| \le \frac{M}{(n+1)!} |x - x_0|^{n+1} \quad \text{where } -M \le f^{(n+1)}(c) \le M \text{ for all } c \text{ in } I$$

Assignment

- Mastery Quiz 8 will be distributed at the end of today's lab. For this quiz, you will need to find a Taylor polynomial and an estimate for the size of the remainder.
- Project 2 will be distributed during next week's lab meeting. The completed project will be due the following week, two weeks from today.

Maple Essentials

- The *Taylor Approximation* tutor can be started from the Tools menu:

  Tools → Tutors → Calculus - Single Variable → Taylor Approximation ...

- New Maple commands introduced in this lab include:

Command	Description
`TaylorApproximation`	> `with( Student[Calculus1] ):` explicit Taylor polynomial: > `TaylorApproximation( x*exp(-x), x=0, order=3 );` plot of Taylor polynomial: > `TaylorApproximation( x*exp(-x), x=0, order=3, -1..4, output=plot );`
`add`	computes a finite sum > `add( k^2, k=1..5 )` expands to $1^2+2^2+3^2+4^2+5^2$ (which simplifies to 55) > `add( x^k, k=1..5 )` expands to $x+x^2+x^3+x^4+x^5$
`$`	repetition operator > `x$4` expands to x, x, x, x > `x^k $ k=2..4` expands to x^2, x^3, x^4

Activities

1. Find, and plot, the Maclaurin polynomials of orders $n = 0$, 1, 2, 3, 4, 6, 8, and 10 for each of the following functions:

   (i) e^{-x} (ii) $\cos(\pi x)$ (iii) $\ln(1+x)$ (iv) $x\sin(x)$

2. Find, and plot, the Taylor polynomials of orders $n = 0$, 1, 2, 3, 4, 6, 8, and 10 about $x = x_0$ for each of the following pairs of f and x_0:

   (i) $f(x) = e^x$, $x_0 = 1$ (ii) $f(x) = e^{-x}$, $x_0 = \ln 2$
   (iii) $f(x) = \frac{1}{x}$, $x_0 = -1$ (iv) $f(x) = \frac{1}{x+2}$, $x_0 = 3$
   (v) $f(x) = \sin(\pi x)$, $x_0 = 1/2$ (vi) $f(x) = \cos x$, $x_0 = \pi/2$

3. Find an interval I containing $x_0 = 0$ over which $f(x)$ can be approximated by $p(x)$ to (at least) three decimal-place accuracy throughout the interval. Find M that satisfies the conditions of the Remainder Estimation Theorem.

   (i) $f(x) = e^x$, $p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
   (ii) $f(x) = \cos x$, $p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
   (iii) $f(x) = \ln(1+x)$, $p(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$
   (iv) $f(x) = \cos(2x)$, $p(x) = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6$

   Steps:

   (a) Plot the function $|R_n(x)|$ with the line $y = 0.0005$ over an interval containing $x_0 = 0$.
   (b) Use the `fsolve` command with bounds to find the intersection points. This will be your interval I.
   (c) Find the maximum value of $|f^{(n+1)}(c)|$ for all c in I. This will be M.
   (d) Verify that the Remainder Estimation Theorem is satisfied.

Additional Note

The following explicit Maple commands can be used to find the n^{th} Taylor polynomial of f at $x = x_0$.

```
> f:=cos(2*x); x0:=Pi/4; n:=20;             # define the function, base point, and order
> pn:=eval(f,x=x0) + add( eval(diff(f,x$k),x=x0)/k!*(x-x0)^k, k=1..n );
```