

## A More Rigorous Approach to Limits

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### Overview

The rigorous  $\epsilon$ - $\delta$  definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts.

### Maple Essentials

- The *EpsilonDelta* maplet is available from the course website:

<http://people.math.sc.edu/calclab/141L-S19/labs/> → EpsilonDelta

### Related course material/Preparation

§2.3 of the textbook. Let us first recall the definition of limit: Let  $f(x)$  be defined for all  $x$  in some open interval containing the number  $a$ , with the possible exception that  $f(x)$  need not be defined at  $a$ . We will write

$$\lim_{x \rightarrow a} f(x) = L$$

if given any number  $\epsilon > 0$  we can find a number  $\delta > 0$  such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta.$$

In general,  $\epsilon$  and  $\delta$  are meant to be very small numbers. Therefore, intuitively, the definition states that  $f(x)$  will be very close to  $L$  that is,  $|f(x) - L| < \epsilon$ , when  $x$  is very close to  $a$  ( $|x - a| < \delta$ ). The task is to show that, for any given  $\epsilon$  (no matter how close  $f(x)$  is to  $L$ ), you can always find a  $\delta$ -needed closeness of  $x$  to  $a$ -to make it work.

### Activities

From our discussion, our job is to find a  $\delta$  for a given  $\epsilon$  such that, when  $a - \delta < x < a + \delta$ , the inequality  $|f(x) - L| < \epsilon$  holds. Therefore, we need to solve for a range  $a - \delta < x < a + \delta$  of  $x$  from the given inequality  $|f(x) - L| < \epsilon$ . Ideally, we would like to find a formula of  $\delta$  in terms of  $\epsilon$  (see examples 2 and 5 of §2.3) that will work for any given  $\epsilon$ . However, such formulas are in general very hard to find. Moreover, the value of  $\delta$  is not unique, as any value that is smaller than a solution would work, too. For each of the limits below, we will use Maple's **solve** command to help us to find the largest  $\delta$  that works for the given  $\epsilon$  and the interactive *EpsilonDelta* maplet provides a tool to visualize relations between  $\delta$  and  $\epsilon$ .

(Follow the General Directions on the back of this page.)

1.  $\lim_{x \rightarrow 9} \sqrt{x} = 3$ ,  $\epsilon = 0.15$ ,  $\epsilon = 0.05$
2.  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6$ ,  $\epsilon = 0.2$ ,  $\epsilon = 0.05$
3.  $\lim_{x \rightarrow 3} (5x - 2) = 13$ ,  $\epsilon = .05$ ,  $\epsilon = .01$

4.  $\lim_{x \rightarrow 2} (x^2 + 3x - 1) = 9$ ,  $\epsilon = 0.8$ ,  $\epsilon = 0.6$

HINT: Since we also have  $\lim_{x \rightarrow -5} (x^2 + 3x - 1) = 9$ , Maple's `solve` command will return extra solutions. Which interval should you choose for problem 4?

### Use Maple's `solve` command to solve inequalities

Maple's `solve` command was introduced in the Lab 4 to solve equations. It can also be used to solve inequalities. We will input most of our inequalities as follows:

`> solve(abs(f(x)-L) < epsilon, x);`

For example, if we want to know where  $|\sqrt{x} - 2| < 0.05$  we would use the following command

`> solve(abs(sqrt(x)-2) < 0.05, x);`

and Maple would return the interval (3.8025, 4.2025) as the solution (your TA will explain Maple's notation)

### General Directions

1. Look at the limit and identify  $f(x)$ ,  $L$ ,  $a$ , and  $\epsilon$ .
2. Launch the *EpsilonDelta* maplet and click **Modify or Make Your Own Problem**. Enter the function  $f(x)$ ,  $a$ ,  $L$ , and  $\epsilon$ .
3. Click **Save Problem and Close**. You should see the graph of  $f(x)$  in blue with a cyan vertical stripe that goes from  $a - \delta$  to  $a + \delta$  and a pink horizontal stripe that goes from  $L - \epsilon$  to  $L + \epsilon$ . You should also see a brown rectangle extends vertically from the smallest value of  $f(x)$  to the largest value of  $f(x)$  for  $x$  from  $a - \delta$  to  $a + \delta$ . You may change the size of this rectangle by changing the value of  $\delta$ , which can be done using the slider (for  $0.1 \leq \delta \leq 1$ ) or by typing in (any value).
4. Your task is to determine the largest value of  $\delta$  that keeps the brown rectangle completely inside the pink stripe. You can use **Zoom In** to increase the accuracy.
5. When you think you are done, record your final value of  $\delta$ .
6. Now we will find the value of  $\delta$  more precisely using Maple's `solve` command.
7. Use the arrow notation (`:=x->`) to define the function  $f(x)$ . Use `:=` to assign  $L$ ,  $a$ , and `epsilon` to their respective values.
8. Use the `solve` command as follows  
`> solve(abs(f(x) - L) < epsilon, x);`  
 Maple should return an interval or intervals.
9. Choose the interval that contains  $a$ . Find the distances from  $a$  to the left bound and from  $a$  to the right bound of the interval (both of them should be positive.) The *smallest* of these two values is the *largest*  $\delta$  that works for this  $\epsilon$ .
10. Your values from the *EpsilonDelta* maplet and from using the `solve` command should be very close.

### Remark:

For some simple functions like linear functions, `solve` can be used to find general formulas of  $\delta$  in term of  $\epsilon$ . Try the following and compare it to problem 3:

`solve(abs(5*x-2-13)<epsilon,x) assuming epsilon>0;`

### Assignment

Review Lab 1 to Lab 5 for Lab Quiz 1 next week and your lab instructor may give other assignment for each section.