

A More Rigorous Approach to Limits

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Overview

The rigorous ϵ - δ definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts.

Maple Essentials

- The *EpsilonDelta* maplet is available from the course website:

<http://people.math.sc.edu/calclab/141L-S17/labs/> → EpsilonDelta

Related course material/Preparation

§2.3 of the textbook. Let us first recall the definition of limit: Let $f(x)$ be defined for all x in some open interval containing the number a , with the possible exception that $f(x)$ need not be defined at a . We will write

$$\lim_{x \rightarrow a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - a| < \delta.$$

In general, ϵ and δ are meant to be very small numbers. Therefore, intuitively, the definition states that $f(x)$ will be very close to L that is, $|f(x) - L| < \epsilon$, when x is very close to a ($|x - a| < \delta$). The task is to show that, for any given ϵ (no matter how close $f(x)$ is to L), you can always find a δ -needed closeness of x to a -to make it work.

Activities

From our discussion, our job is to find a δ for a given ϵ such that, when $a - \delta < x < a + \delta$, the inequality $|f(x) - L| < \epsilon$ holds. Therefore, we need to solve for a range $a - \delta < x < a + \delta$ of x from the given inequality $|f(x) - L| < \epsilon$. Ideally, we would like to find a formula of δ in terms of ϵ (see examples 2 and 5 of §2.3) that will work for any given ϵ . However, such formulas are in general very hard to find. Moreover, the value of δ is not unique, as any value that is smaller than a solution would work, too. For each of the limits below, we will use Maple's `solve` command to help us to find the largest δ that works for the given ϵ and the interactive *EpsilonDelta* maplet provides a tool to visualize relations between δ and ϵ .

(Follow the General Directions on the back of this page.)

1. $\lim_{x \rightarrow 9} \sqrt{x} = 3$, $\epsilon = 0.15$, $\epsilon = 0.05$
2. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6$, $\epsilon = 0.2$, $\epsilon = 0.05$
3. $\lim_{x \rightarrow 3} (5x - 2) = 13$, $\epsilon = .05$, $\epsilon = .01$

4. $\lim_{x \rightarrow 2} (x^2 + 3x - 1) = 9$, $\epsilon = 0.8$, $\delta = 0.6$
 HINT: Since we also have $\lim_{x \rightarrow -5} (x^2 + 3x - 1) = 9$, Maple's `solve` command will return extra solutions. Which interval should you choose for problem 4?

Use Maple's `solve` command to solve inequalities

Maple's `solve` command was introduced in the Lab 4 to solve equations. It can also be used to solve inequalities. We will input most of our inequalities as follows:

`> solve(abs(f(x)-L) < epsilon, x);`

For example, if we want to know where $|\sqrt{x} - 2| < 0.05$ we would use the following command

`> solve(abs(sqrt(x)-2) < 0.05, x);`

and Maple would return the interval (3.8025,4.2025) as the solution (your TA will explain Maple's notation)

General Directions

1. Look at the limit and identify $f(x)$, L , a , and ϵ .
2. Launch the *EpsilonDelta* maplet and click **Modify or Make Your Own Problem**. Enter the function $f(x)$, a , L , and ϵ .
3. Click **Save Problem and Close**. You should see the graph of $f(x)$ in blue with a cyan vertical stripe that goes from $a - \delta$ to $a + \delta$ and a pink horizontal stripe that goes from $L - \epsilon$ to $L + \epsilon$. You should also see a brown rectangle extends vertically from the smallest value of $f(x)$ to the largest value of $f(x)$ for x from $a - \delta$ to $a + \delta$. You may change the size of this rectangle by changing the value of δ , which can be done using the slider (for $0.1 \leq \delta \leq 1$) or by typing in (any value).
4. Your task is to determine the largest value of δ that keeps the brown rectangle completely inside the pink stripe. You can use **Zoom In** to increase the accuracy.
5. When you think you are done, record your final value of δ .
6. Now we will find the value of δ more precisely using Maple's `solve` command.
7. Use the arrow notation (`:=x->`) to define the function $f(x)$. Use `:=` to assign L , a , and `epsilon` to their respective values.
8. Use the `solve` command as follows
`> solve(abs(f(x) - L) < epsilon, x);`
 Maple should return an interval or intervals.
9. Choose the interval that contains a . Find the distances from a to the left bound and from a to the right bound of the interval (both of them should be positive.) The *smallest* of these two values is the *largest* δ that works for this ϵ .
10. Your values from the *EpsilonDelta* maplet and from using the `solve` command should be very close.

Remark:

For some simple functions like linear functions, `solve` can be used to find general formulas of δ in term of ϵ . Try the following and compare it to problem 3:

`solve(abs(5*x-2-13)<epsilon,x) assuming epsilon>0;`

Assignment

Review Lab 1 to Lab 5 for Lab Quiz 1 next week and your lab instructor may give other assignment for each section.